

LIMIT THEORY FOR THE SAMPLE AUTOCOVARIANCE FOR HEAVY TAILED STATIONARY INFINITELY DIVISIBLE PROCESSES GENERATED BY CONSERVATIVE FLOWS

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ABSTRACT. This study aims to develop the limit theorems on the sample autocovariances and sample autocorrelations for certain stationary infinitely divisible processes. We consider the case where the infinitely divisible process has heavy tail marginals and is generated by a conservative flow. Interestingly, the growth rate of the sample autocovariances is determined by not only heavy tailedness of the marginals but also memory length of the process. Although this feature was first observed by Resnick et al. (2000) for some very specific processes, we will propose a more general framework from the viewpoint of infinite ergodic theory. Consequently, the asymptotics of the sample autocovariances can be more comprehensively discussed.

1. INTRODUCTION

For a discrete stationary process $(X_n, n \geq 0)$, the sample autocovariance function and the sample autocorrelation function are vital statistics in the analysis of dependence structure of the process. According to Wold decomposition (see p. 187 in Brockwell and Davis (1991)), every stationary process with zero mean and finite variance can be represented by the sum of an infinite-order moving average and a perfectly predictable process. This fact justifies, to some extent, that every stationary process of finite second moment can be approximated by a moving average process (or equivalently, an $\text{ARMA}(p, q)$ process of finite order). Thus, in a classical L^2 -context, linear models are sufficient for data analysis; indeed, the sample autocorrelation function has traditionally been an important model-fitting and diagnostic tools (see, for example, Chapter 7 of Brockwell and Davis (1991)).

If stationary processes lack finite variance, they cannot generally be approximated by linear processes. Thus, it is natural to question whether classical methods based on sample autocorrelations are still plausible. For instance, a major feature of heavy tail models is that the sample autocorrelation converges to a *random* limit. The presence of random limits means that traditional model-fitting and diagnostic tools such as the Akaike Information Criterion or Yule-Walker estimators may

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yield erroneous results. For more details, see Davis and Resnick (1996), Resnick and Van Den Berg (2000) and Resnick et al. (1999).

To determine the limit behavior of the sample autocovariances of infinite variance stationary processes, it is also important to see how rapidly the sample autocovariances grow. Many studies have revealed that if the tail of a marginal distribution is regularly varying with index $-\alpha$ for some $0 < \alpha < 2$, then a proper normalizing sequence (c_n) for the sample autocovariances may be written as $c_n = n^{1-2/\alpha}L(n)$, where $L(n)$ is a slowly varying function. Among the processes that possess such type of normalizing sequence are the linear process whose noise distribution has a balanced regularly varying tail (Davis and Resnick (1986)), the bilinear process (Davis and Resnick (1996), Resnick and Van Den Berg (2000)), certain ARCH processes (Davis and Mikosch (1998)) and α -stable moving average processes (Resnick et al. (1999)).

Resnick et al. (2000) reported an interesting phenomenon with respect to the growing rate of the sample autocovariance. They considered a process of the form

$$X_n = \int_{\mathbb{Z}^{\mathbb{N}}} f \circ T^n(x) dM(x),$$

where M is a symmetric α -stable random measure defined on $(\mathbb{Z}^{\mathbb{N}}, \mathcal{B}(\mathbb{Z}^{\mathbb{N}}))$, and $T(x_0, x_1, \dots) = (x_1, x_2, \dots)$ is the left shift map defined on $\mathbb{Z}^{\mathbb{N}}$. Furthermore, M is assumed to have a control measure of the form $\mu(A) = \sum_{i \in \mathbb{Z}} \pi_i P_i(A)$, where $P_i(\cdot)$ is a probability law of an irreducible, null recurrent Markov chain with state space \mathbb{Z} , and (π_i) is its unique (up to multiplicative factors) σ -finite and invariant measure. By introducing an extra parameter $0 \leq \beta \leq 1$, they proved that a proper normalizing sequence in this situation is $c_n = n^{(1-\beta)(1-\alpha/2)}L(n)$. The parameter β accounts for the significantly longer memory of this process, relative to the other processes described in the previous paragraph; more details can be found in Samorodnitsky (2005).

An obvious drawback of the process studied by Resnick et al. (2000) is the highly specific form of the process and its control measure. In this paper, we propose a more general framework inspired by the infinite ergodic theory, in which the asymptotics of the sample autocovariances can be more comprehensively assessed. In terms of the growth rate of the sample autocovariance and its weak limit, we will demonstrate that results similar to those of Resnick et al. (2000) are obtainable in the generalized framework.

2. THE SETUP

We consider an infinitely divisible process

$$(2.1) \quad X_n = \int_E f \circ T^n(x) dM(x), \quad n = 1, 2, \dots,$$

where M is an independently scattered infinitely divisible random measure on a measurable space (E, \mathcal{E}) and $f : E \rightarrow \mathbb{R}$ is a deterministic function, and $T : E \rightarrow E$ is a measurable map. We often denote $f_n(x) = f \circ T^n(x)$. The random measure M is assumed to be homogeneous symmetric and have a local Lévy measure ρ and a σ -finite *infinite* control measure μ . We assume, throughout the paper, that a Gaussian component is identically zero. By these assumptions on the random measure M , we may write, for every $A \in \mathcal{E}$ of finite μ -measure,

$$Ee^{iuM(A)} = \exp \left\{ -\mu(A) \int_{\mathbb{R}} (1 - \cos(ux)) \rho(dx) \right\} \quad u \in \mathbb{R}.$$

A measurable function f is assumed to have a support of finite μ -measure. Moreover, we assume that

$$(2.2) \quad f \in L^2(\mu) \quad \text{with} \quad \mu(f^2) = \int_E f(x)^2 \mu(dx) > 0.$$

One of the central assumption in our work is related to the heavy tailedness of the process $\mathbf{X} = (X_1, X_2, \dots)$. We assume that ρ has a regularly varying tail with index $-\alpha$, $0 < \alpha < 2$,

$$(2.3) \quad \rho(\cdot, \infty) \in RV_{-\alpha} \quad \text{at infinity.}$$

From now, we express ρ by ρ_α , emphasizing its dependence on the tail parameter α .

Assuming that $T : E \rightarrow E$ preserves the control measure μ , the process $\mathbf{X} = (X_1, X_2, \dots)$ turns out to be a well-defined stationary infinitely divisible process with the Lévy measure of each X_n given by

$$(\rho_\alpha \times \mu)\{(x, s) : xf_n(s) \in \cdot\}.$$

See Rajput and Rosiński (1989) for further information on spectral representations of infinitely divisible processes. The assumptions on the local Lévy measure ρ_α and $f \in L^2(\mu)$ imply that

$$(\rho_\alpha \times \mu)\{(x, s) : xf_n(s) > \lambda\} \sim \left(\int_E |f(s)|^\alpha \mu(ds) \right) \rho_\alpha(\lambda, \infty) \quad \text{as } \lambda \rightarrow \infty.$$

We can see from Rosiński and Samorodnitsky (1993) that the tail of X_n is asymptotically the same as the tail of the Lévy measure. That is,

$$P(X_n > \lambda) \sim \left(\int_E |f(s)|^\alpha \mu(ds) \right) \rho_\alpha(\lambda, \infty) \quad \text{as } \lambda \rightarrow \infty.$$

Due to a regularly varying tail of ρ_α , this implies that the process $\mathbf{X} = (X_1, X_2, \dots)$ belongs to the domain of attraction of a S α S law.

The other crucial assumption in this paper is that the process \mathbf{X} is generated by a conservative flow. This assumption is known to be related to long memory in the process \mathbf{X} ; the length of memory observed in \mathbf{X} is significantly longer than that in the process generated by a dissipative flow (e.g. α -stable moving average processes). See for example, Samorodnitsky (2004) and Roy (2008). The assumptions on the flow and regular variation of the tail of the local Lévy measure

entirely characterize the limit theorems on the sample autocovariances. We will see later that the length of memory can be parameterized by some ergodic theoretical notion imposed on the flow.

3. ERGODIC THEORETICAL NOTIONS

In this section, we will present the basic notions on ergodic theory used in the sequel. For further studies, the main references are Krengel (1985), Aaronson (1997), and Zweimüller (2009).

Let (E, \mathcal{E}, μ) be a σ -finite, infinite measure space. We will often denote $A = B \bmod \mu$ for $A, B \in \mathcal{E}$ when $\mu(A \triangle B) = 0$.

Let $T : E \rightarrow E$ be a measurable map. T is called *ergodic* if any (almost) invariant set A with respect to T (i.e., $T^{-1}A = A \bmod \mu$) satisfies $\mu(A) = 0$ or $\mu(A^c) = 0$.

The map T is said to be *conservative* if

$$\sum_{n=1}^{\infty} \mathbf{1}_A \circ T^n = \infty \quad \text{a.e. on } A$$

for any $A \in \mathcal{E}$, $0 < \mu(A) < \infty$. When the whole sequence (T^n) gets involved, (T^n) is particularly called a *flow*.

In view of the Hopf decomposition (see Krengel (1985)), any state space E can be partitioned into two measurable invariant subsets C and D , such that the map T is conservative on C and $D = E \setminus C$. We usually refer to C and D as a conservative part and a dissipative part, respectively. From its definition, C is viewed as a set such that, departing from an arbitrary $A \subseteq C$, one could keep coming back to A infinitely often. On the contrary, even if starting from $A \subseteq D$, one may not come back to A quite often.

Next we define a *dual operator* $\widehat{T} : L^1(\mu) \rightarrow L^1(\mu)$ by

$$\widehat{T}f = \frac{d(\nu_f \circ T^{-1})}{d\mu},$$

where ν_f is a signed measure defined by $\nu_f(A) = \int_A f d\mu$, $A \in \mathcal{E}$. It is worth providing the dual relation

$$(3.1) \quad \int_E \widehat{T}f \cdot g d\mu = \int_E f \cdot g \circ T d\mu$$

for $f \in L^1(\mu)$, $g \in L^\infty(\mu)$. Note that, for any nonnegative measurable function f on E , a similar definition gives a nonnegative measurable function $\widehat{T}f$, and that (3.1) holds for any two nonnegative measurable functions f and g .

A conservative ergodic and measure preserving map T is said to be *pointwise dual ergodic*, if there exists a normalizing sequence $a_n \nearrow \infty$ such that

$$(3.2) \quad \frac{1}{a_n} \sum_{k=1}^n \widehat{T}^k f \rightarrow \mu(f) \quad \text{a.e. for every } f \in L^1(\mu).$$

We often require that the above convergence takes place uniformly on a set of finite measure. Let $A \in \mathcal{E}$ with $0 < \mu(A) < \infty$. A is said to be a uniform set for a conservative ergodic and measure preserving map T , if there exist a normalizing sequence $a_n \nearrow \infty$ and a nonnegative measurable function $f \in L^1(\mu)$ such that

$$(3.3) \quad \frac{1}{a_n} \sum_{k=1}^n \widehat{T}^k f \rightarrow \mu(f) \quad \text{uniformly, a.e. on } A.$$

If a measurable function f in (3.3) can be replaced by an indicator function $\mathbf{1}_A$, the set A is particularly called a *Darling-Kac set*. From the similar argument as the proof of Proposition 3.7.5 in Aaronson (1997), one can see that T is pointwise dual ergodic if and only if T admits a uniform set. It is important to note that it is legitimate to use the same sequence (a_n) both in (3.2) and (3.3).

We often need to put a more strict assumption than (3.3). Let $A \in \mathcal{E}$ with $0 < \mu(A) < \infty$. A is said to be a *uniformly returning set* for a conservative ergodic and measure preserving map T , if there exist a normalizing sequence $b_n \nearrow \infty$ and a nonnegative measurable function $f \in L^1(\mu)$ such that

$$(3.4) \quad b_n \widehat{T}^n f \rightarrow \mu(f) \quad \text{uniformly, a.e. on } A.$$

Clearly any uniformly returning set is a uniform set. Further information on uniformly returning sets is given, for example, in Kesseböhmer and Slassi (2007).

Given a uniform set (or a Darling-Kac set or a uniformly returning set) A , a natural question is how often the set A will be visited as we evolve along the flow (T^n) . Such frequency is usually measured by a wandering rate

$$(3.5) \quad w_n = \mu\left(\bigcup_{k=0}^{n-1} T^{-k} A\right).$$

There are some other alternative expressions for (3.5). To get those alternatives, we define the first entrance time to A

$$\varphi(x) = \min\{n \geq 1 : T^n x \in A\}.$$

(Notice that $\varphi < \infty$ a.e. on E , if T is conservative ergodic and measure preserving.) It is elementary to prove that $\mu(A \cap \{\varphi > k\}) = \mu(A^c \cap \{\varphi = k\})$, $k \geq 1$. Therefore, we get

$$(3.6) \quad w_n = \mu(A) + \sum_{k=1}^{n-1} \mu(A^c \cap \{\varphi = k\}) = \sum_{k=0}^{n-1} \mu(A \cap \{\varphi > k\}).$$

This, in turn, implies

$$w_n \sim \mu(\varphi < n) \quad \text{as } n \rightarrow \infty.$$

Let T be a conservative ergodic and measure preserving map. Let A be a uniform set determined by T . Then there is a precise connection between the return sequence (w_n) and the normalizing sequence (a_n) in (3.3) (and, hence, also in (3.2)), if regular variation is assumed. Specifically, if either $(w_n) \in RV_{1-\beta}$ or $(a_n) \in RV_\beta$ for some $\beta \in [0, 1]$, then

$$(3.7) \quad a_n \sim \frac{1}{\Gamma(2-\beta)\Gamma(1+\beta)} \frac{n}{w_n} \quad \text{as } n \rightarrow \infty.$$

Proposition 3.8.7 in Aaronson (1997) gives one direction of this statement, but the argument is easily reversed.

Analogously, a similar kind of connection between (w_n) and (b_n) in (3.4) was shown by Kesseböhmer and Slassi (2007). If either $(w_n) \in RV_{1-\beta}$ or $(b_n) \in RV_{1-\beta}$ for some $\beta \in (0, 1]$, then

$$(3.8) \quad b_n \sim \Gamma(\beta)\Gamma(2-\beta)w_n.$$

4. LIMIT THEOREM ON THE SAMPLE AUTOCOVARIANCES

This section presents the main limit theorems on the sample autocovariances and the sample autocorrelations. Let T be a conservative ergodic and measure preserving map on a σ -finite infinite measure space (E, \mathcal{E}, μ) . Furthermore, T is pointwise dual ergodic and, hence, T admits some uniform set $A \in \mathcal{E}$ with $0 < \mu(A) < \infty$. We suppose that the normalizing sequence (a_n) for pointwise dual ergodicity is regularly varying with exponent $0 \leq \beta < 1$. As assumed in Section 2, M is a symmetric homogeneous infinitely divisible random measure on (E, \mathcal{E}) with control measure μ and local Lévy measure ρ_α , which satisfies (2.3).

We will add an extra assumption on the lower tail of ρ_α : for some $p_0 \in (0, 2)$,

$$(4.1) \quad x^{p_0} \rho_\alpha(x, \infty) \rightarrow 0 \quad \text{as } x \downarrow 0.$$

We will assume that $f : E \rightarrow \mathbb{R}$ satisfies integrability condition (2.2) and that f is supported by the uniform set A .

As we will see in Section 5, a certain operator T (e.g., Markov shift operator; see Chapter 4 in Aaronson (1997)) satisfies

$$\widehat{T}^k \mathbf{1}_{A^c \cap \{\varphi=k\}}(x) = \mu(A^c \cap \{\varphi=k\}), \quad \text{for all } x \in A, \quad k \geq 1.$$

We generalize this property by assuming

$$(4.2) \quad \frac{1}{\mu(\varphi \leq n)} \sum_{k=1}^n \widehat{T}^k \mathbf{1}_{A^c \cap \{\varphi=k\}}(x) \quad \text{is uniformly bounded on } A.$$

We first want to define some normalizing sequence (c_n) , which can capture how rapidly the sample autocovariances of the process \mathbf{X} given in (2.1) grow. Let $U_\alpha(x) = \rho_\alpha(x, \infty)$, $x > 0$. We

define the right continuous inverse of $U_\alpha(x)$ by

$$U_\alpha^\leftarrow(y) = \inf\{x > 0 : U_\alpha(x) \leq y\}, \quad y > 0.$$

Given the normalizing sequence (a_n) for pointwise dual ergodicity and its wandering rate sequence (w_n) , we define

$$(4.3) \quad c_n = 2^{2/\alpha} C_{\alpha,\beta} C_{\alpha/2}^{-2/\alpha} a_n (U_\alpha^\leftarrow(w_n^{-1}))^2,$$

where

$$C_{\alpha,\beta} = \Gamma(1 + \beta) (EM_\beta(1 - V_\beta)^{\alpha/2})^{2/\alpha}$$

(the definition of $M_\beta(t)$ and V_β are given in the Appendix). On the other hand, $C_{\alpha/2}$ is a tail constant for an $\alpha/2$ -stable random variable; see Samorodnitsky and Taqqu (1994). It then follows from Proposition 4.2 below that (c_n) satisfies the asymptotic relation

$$(4.4) \quad \rho_\alpha((c_n a_n^{-1})^{1/2}, \infty) \sim 2^{-1} C_{\alpha/2} (\mu(f^2) a_n)^{\alpha/2} \left(\int_E \left| \sum_{k=1}^n f_k(x)^2 \right|^{\alpha/2} \mu(dx) \right)^{-1} \text{ as } n \rightarrow \infty.$$

From the definition (4.3), it is easy to obtain the regular variation exponent for (c_n) :

$$(4.5) \quad c_n \in RV_{\beta+2(1-\beta)/\alpha}.$$

Therefore, the growth rate of the sample autocovariance of the process \mathbf{X} is determined by not only heavy tailedness of the marginals but also the length of memory. This is in contrast to the case of the processes generated by dissipative flows, e.g., α -stable moving averages studied by Resnick et al. (1999), where it was shown that the sample autocovariances of the α -stable moving averages grow at a regularly varying rate with exponent $2/\alpha$. A substitution of $\beta = 0$ into (4.5) yields $c_n \in RV_{2/\alpha}$, which implies that $\beta = 0$ corresponds to the shortest memory in the process \mathbf{X} . As β gets closer to 1, it is expected to exhibit longer memory.

Our argument for the main limit theorems on the sample autocovariances will be separated into two cases. First, we discuss the case where α and β lie in the range

$$(4.6) \quad \text{either } 1 < \alpha < 2, 0 \leq \beta < 1 \text{ or } 0 < \alpha \leq 1, 0 \leq \beta < 1/(2 - \alpha).$$

In this case, the main theorem will be proved by a series of propositions (Propositions 4.3 - 4.7), together with Lemma 6.4. If α and β lie outside the range (4.6), in addition to the abovementioned propositions, we will apply Lemma 6.5 or 6.6.

Finally, we denote the sample autocovariance of the process \mathbf{X} by

$$\hat{\gamma}_n(h) = \frac{1}{n} \sum_{k=1}^n X_k X_{k+h}, \quad h = 0, 1, 2, \dots,$$

and the sample autocorrelation function by $\hat{\rho}_n(h) = \hat{\gamma}_n(h)/\hat{\gamma}_n(0)$, $h = 0, 1, 2, \dots$.

Theorem 4.1. *Let T be a conservative ergodic and measure preserving map on a σ -finite infinite measure space (E, \mathcal{E}, μ) . We assume that T is a pointwise dual ergodic map with normalizing sequence $(a_n) \in RV_\beta$. Suppose that T admits a uniform set $A \in \mathcal{E}$, $0 < \mu(A) < \infty$, and (4.2) is fulfilled.*

Let M be a symmetric homogeneous infinitely divisible random measure on (E, \mathcal{E}) with control measure μ and local Lévy measure ρ_α , which satisfies (2.3) and (4.1).

Let $f : E \rightarrow \mathbb{R}$ be a measurable function that is supported by the uniform set A and satisfies integrability condition (2.2).

Let α and β lie in the range (4.6). Then, the stationary infinitely divisible process \mathbf{X} given in (2.1) satisfies for $H \geq 0$,

$$(4.7) \quad \left(\frac{n}{c_n} \widehat{\gamma}_n(h), h = 0, \dots, H \right) \Rightarrow (\mu(f \cdot f_h)W, h = 0, \dots, H).$$

Here, W is a positive strictly stable random variable of exponent $\alpha/2$, i.e., the characteristic function of W is given by

$$(4.8) \quad Ee^{iuW} = \exp\left\{ \int_{(0, \infty)} (e^{iux} - 1) \rho_*(dx) \right\} \quad u \in \mathbb{R},$$

with $\rho_(dx) = 2^{-1} \alpha C_{\alpha/2} x^{-1-\alpha/2} dx$, $x > 0$.*

As a consequence, we also get

$$(4.9) \quad \widehat{\rho}_n(h) \xrightarrow{p} \frac{\mu(f \cdot f_h)}{\mu(f^2)}.$$

On the other hand, if α and β lie outside the range (4.6), we additionally suppose either (i) or (ii) below.

(i): $T \times T$ is still a conservative and ergodic map on $(E \times E, \mathcal{E} \times \mathcal{E}, \mu \times \mu)$. Moreover, $T \times T$ is a pointwise dual ergodic map with normalizing sequence $(a'_n) \in RV_{2\beta-1}$, and further, we extend the condition (4.2) to a two-dimensional version:

$$(4.10) \quad \frac{1}{(\mu \times \mu)(\varphi(x, y) \leq n)} \sum_{k=1}^n (\widehat{T \times T})^k \mathbf{1}_{(A \times A)^c \cap \{\varphi(x, y) = k\}} \quad \text{is uniformly bounded on } A \times A,$$

where $\varphi(x, y) = \min\{n \geq 1 : (T^n x, T^n y) \in A \times A\}$ is the first entrance time to the set $A \times A$, and $\widehat{T \times T}$ is a dual operator for $T \times T$.

(ii): A is a uniformly returning set for $\mathbf{1}_A$, i.e., there exists an increasing normalizing sequence (b_n) such that

$$(4.11) \quad b_n \widehat{T}^n \mathbf{1}_A \rightarrow \mu(A) \quad \text{uniformly, a.e. on } A.$$

Moreover, f is a bounded function.

Then, (4.7) and (4.9) follow again.

We will first start by checking how rapidly the sample autocovariance of the process \mathbf{X} grows when it is associated with a conservative flow.

Proposition 4.2. *Under the assumptions of Theorem 4.1,*

$$\left(\int_E |S_n(f^2)|^{\alpha/2} d\mu \right)^{2/\alpha} \sim \mu(f^2) C_{\alpha,\beta} a_n w_n^{2/\alpha},$$

where

$$C_{\alpha,\beta} = \Gamma(1 + \beta) (EM_\beta(1 - V_\beta)^{\alpha/2})^{2/\alpha}$$

(the definition of $M_\beta(t)$ and V_β are given in the Appendix).

Proof. We write

$$\left(\int_E |S_n(f^2)|^{\alpha/2} d\mu \right)^{2/\alpha} = a_n \mu(\varphi \leq n)^{2/\alpha} \left(\int_E \left| \frac{S_n(f^2)}{a_n} \right|^{\alpha/2} d\mu_n \right)^{2/\alpha},$$

where $\mu(\cdot) = \mu(\cdot \cap \{\varphi \leq n\}) / \mu(\varphi \leq n)$. Because of (3.6) and (3.7),

$$\sup_{n \geq 1} \int_E \left| \frac{S_n(f^2)}{a_n} \right| d\mu_n \leq \sup_{n \geq 1} \frac{n}{a_n \mu(\varphi \leq n)} \int_E f^2 d\mu < \infty.$$

Hence, $(|S_n(f^2)/a_n|^{\alpha/2}, n \geq 1)$ is a uniformly integrable sequence with respect to μ_n . Therefore, Lemma 6.2 in the Appendix implies

$$\left(\int_E \left| \frac{S_n(f^2)}{a_n} \right|^{\alpha/2} d\mu_n \right)^{2/\alpha} \rightarrow \mu(f^2) C_{\alpha,\beta}.$$

□

As a preparation for the proof of Theorem 4.1, we decompose the process \mathbf{X} by the magnitude of the Lévy jumps: we first let

$$\rho_{\alpha,1}(\cdot) = \rho_\alpha(\cdot \cap \{x : |x| > 1\}),$$

$$\rho_{\alpha,2}(\cdot) = \rho_\alpha(\cdot \cap \{x : |x| \leq 1\}).$$

Let M_i , $i = 1, 2$, denote homogeneous symmetric infinitely divisible random measures with the same control measure μ and local Lévy measures $\rho_{\alpha,i}$, $i = 1, 2$.

Then, X_n can be written as

$$X_n \stackrel{d}{=} \int_E f_n(x) dM_1(x) + \int_E f_n(x) dM_2(x).$$

Denote $X_n^{(i)} = \int_E f_n(x) dM_i(x)$, $i = 1, 2$. We may write

$$n\widehat{\gamma}_n(h) = \sum_{k=1}^n X_k X_{k+h} \stackrel{d}{=} \sum_{k=1}^n X_k^{(1)} X_{k+h}^{(1)} + \sum_{k=1}^n X_k^{(1)} X_{k+h}^{(2)} + \sum_{k=1}^n X_k^{(2)} X_{k+h}^{(1)} + \sum_{k=1}^n X_k^{(2)} X_{k+h}^{(2)}.$$

The main tool used in our proof is a certain series representation of (X_n) , which was developed by Rosiński (1990). We also refer to Section 3.10 in Samorodnitsky and Taqqu (1994). Since μ is a σ -finite measure, one can find an μ -equivalent probability measure μ_0 such that

$$\mu_0(B) = \int_B q(x) \mu(dx),$$

where q is a positive measurable function on E . For $l = 1, 2$, we write $U_{\alpha,l}(x) = \rho_{\alpha,l}(x, \infty)$ for $x > 0$ and define the right continuous inverse of $U_{\alpha,l}(x)$ by

$$U_{\alpha,l}^{\leftarrow}(y) = \inf\{x > 0 : U_{\alpha,l}(x) \leq y\}, \quad y > 0.$$

According to Rosiński (1990), $X_n^{(l)}$ can be represented in law as

$$(X_n^{(l)}, n \geq 0) \stackrel{d}{=} \left(\sum_{i=1}^{\infty} \epsilon_i U_{\alpha,l}^{\leftarrow} \left(\frac{\Gamma_i q(V_i)}{2} \right) f_n(V_i), n \geq 0 \right),$$

where (ϵ_i) is an i.i.d. Rademacher sequence taking 1 or -1 with probability $1/2$, Γ_i is the i th jump time of a unit rate Poisson process, and (V_i) is a sequence of i.i.d. random variables with common distribution μ_0 .

In the next proposition, the series representation for $\sum_{k=1}^n X_k^{(l)} X_{k+h}^{(l)}$ will be split into a diagonal part and an off-diagonal part, and the diagonal part can be represented as a specific stochastic integral driven by a positive infinitely divisible random measure.

Proposition 4.3. *For any $H \geq 0$, $n > 0$, and $l = 1, 2$,*

$$\left(\sum_{k=1}^n X_k^{(l)} X_{k+h}^{(l)}, h = 0, \dots, H \right) \stackrel{d}{=} (Y'_{n,l}(h) + Y''_{n,l}(h), h = 0, \dots, H)$$

with

$$Y'_{n,l}(h) = \int_E \sum_{k=1}^n f_k(x) f_{k+h}(x) d\widetilde{M}_l(x),$$

$$Y''_{n,l}(h) = \sum_{i \neq j} \epsilon_i \epsilon_j U_{\alpha,l}^{\leftarrow} \left(\frac{\Gamma_i q(V_i)}{2} \right) U_{\alpha,l}^{\leftarrow} \left(\frac{\Gamma_j q(V_j)}{2} \right) \sum_{k=1}^n f_k(V_i) f_{k+h}(V_j).$$

Here, \widetilde{M}_l is a positive infinitely divisible random measure defined by

$$E e^{iu \widetilde{M}_l(A)} = \exp \left\{ \mu(A) \int_{(0,\infty)} (e^{iux} - 1) \widetilde{\rho}_{\frac{\alpha}{2},l}(dx) \right\}, \quad u \in \mathbb{R},$$

where $\widetilde{\rho}_{\frac{\alpha}{2},l}$ is a local Lévy measure concentrated on the positive half-line such that

$$(4.12) \quad \widetilde{\rho}_{\frac{\alpha}{2},l}(x, \infty) = 2\rho_{\alpha,l}(x^{1/2}, \infty) \quad \text{for } x > 0.$$

Proof. Since

$$\sum_{k=1}^n X_k^{(l)} X_{k+h}^{(l)} \stackrel{d}{=} \sum_{i=1}^{\infty} U_{\alpha,l}^{\leftarrow} \left(\frac{\Gamma_i q(V_i)}{2} \right)^2 \sum_{k=1}^n f_k(V_i) f_{k+h}(V_i) + Y''_{n,l}(h),$$

it suffices to show that the first term converges a.s. and is distributionally equal to $Y'_{n,l}(h)$. For this purpose, from Rosiński (1990), we only check that

$$(4.13) \quad \int_0^\infty P \left(U_{\alpha,l}^\leftarrow \left(\frac{rq(V_1)}{2} \right)^2 \sum_{k=1}^n f_k(V_1) f_{k+h}(V_1) \in \cdot \right) dr = (\tilde{\rho}_{\frac{\alpha}{2},l} \times \mu) \{ (v, x) : v \sum_{k=1}^n f_k(x) f_{k+h}(x) \in \cdot \}$$

and

$$(4.14) \quad \int_E \int_{\mathbb{R}} \min \left(1, \left| v \sum_{k=1}^n f_k(x) f_{k+h}(x) \right| \right) \tilde{\rho}_{\frac{\alpha}{2},l}(dv) \mu(dx) < \infty.$$

Note that the right hand side of (4.13) is exactly the Lévy measure of $Y'_{n,l}(h)$.

Since a simple calculation verifies (4.13), we only prove (4.14). By regular variation of the local Lévy measure ρ_α , the Potter bound (e.g., Proposition 0.8 in Resnick (1987)) provides

$$\tilde{\rho}_{\frac{\alpha}{2},1}(x, \infty) \leq C_1 x^{-(\alpha-\xi)/2}, \quad x > 0$$

for some constants $0 < \xi < \alpha$ and $C_1 > 0$. Also by (4.1), we get an obvious upper bound; for some $C_2 > 0$,

$$\tilde{\rho}_{\frac{\alpha}{2},2}(x, \infty) \leq C_2 x^{-p_0/2}, \quad x > 0.$$

These bounds, together with the fact that f has a support of finite μ -measure and $f \in L^2(\mu)$, can establish (4.14). \square

Now the following expression has been justified:

$$\frac{n}{c_n} \hat{\gamma}_n(h) \stackrel{d}{=} c_n^{-1} \left(Y'_{n,1}(h) + Y''_{n,1}(h) + \sum_{k=1}^n X_k^{(1)} X_{k+h}^{(2)} + \sum_{k=1}^n X_k^{(2)} X_{k+h}^{(1)} + \sum_{k=1}^n X_k^{(2)} X_{k+h}^{(2)} \right).$$

We will describe the idea of the proof of Theorem 4.1. First, we will verify by Proposition 4.4 below that

$$\left(\frac{Y'_{n,1}(h)}{c_n}, h = 0, \dots, H \right) \Rightarrow (\mu(f \cdot f_h)W, h = 0, \dots, H),$$

where W is defined in (4.8). Subsequently, Proposition 4.5 will prove $Y''_{n,1}(h)/c_n \xrightarrow{p} 0$ for every $h \geq 0$. On the other hand, applications of the Cauchy-Schwarz inequality and the result of Proposition 4.3 yield

$$\begin{aligned} \left| \sum_{k=1}^n X_k^{(1)} X_{k+h}^{(2)} \right| &\leq \left(\sum_{k=1}^n (X_k^{(1)})^2 \right)^{1/2} \left(\sum_{k=1}^n (X_{k+h}^{(2)})^2 \right)^{1/2} \\ &\stackrel{d}{=} \left(\sum_{k=1}^n (X_k^{(1)})^2 \right)^{1/2} (Y'_{n,2}(0) + Y''_{n,2}(0))^{1/2}. \end{aligned}$$

Thus, to complete the proof of Theorem 4.1, we need to show $(Y'_{n,2}(0) + Y''_{n,2}(0))/c_n \xrightarrow{p} 0$, which will be established in Propositions 4.6 and 4.7 below. It is noteworthy that Lemma 6.2 in the Appendix

is needed for the proof of Proposition 4.4, while Propositions 4.5 and 4.7 use the results of Lemmas 6.4, 6.5, 6.6, and 6.7.

Proposition 4.4. *For any $H \geq 0$,*

$$\left(\frac{Y'_{n,1}(h)}{c_n}, h = 0, \dots, H \right) \Rightarrow (\mu(f \cdot f_h)W, h = 0, \dots, H),$$

where W is defined in (4.8).

Proof. By virtue of the Cramer-Wold device, we only have to show

$$(4.15) \quad \frac{1}{c_n} \sum_{h=0}^H \theta_h Y'_{n,1}(h) \Rightarrow \sum_{h=0}^H \theta_h \mu(f \cdot f_h)W \quad \text{in } \mathbb{R}$$

for every $\theta_0, \dots, \theta_H \in \mathbb{R}$. Let $\phi(x) = f(x) \sum_{h=0}^H \theta_h f_h(x)$, and denote $S_n(\phi)(x) = \sum_{k=1}^n \phi \circ T^k(x)$. Then, (4.15) is equivalent to

$$(4.16) \quad \frac{1}{c_n} \int_E S_n(\phi)(x) d\widetilde{M}_1(x) \Rightarrow \mu(\phi)W \quad \text{in } \mathbb{R}.$$

A sufficient condition for weak convergence of the left hand side in (4.16) reduces to the following (e.g., Theorem 13.14 in Kallenberg (1997)): for every $r > 0$,

$$(4.17) \quad \int_E \left(\frac{S_n(\phi)}{c_n} \right)^2 \int_0^{rc_n|S_n(\phi)|^{-1}} x \widetilde{\rho}_{\frac{\alpha}{2},1}(x, \infty) dx d\mu \rightarrow \frac{r^{2-\alpha/2} C_{\alpha/2}}{2-\alpha/2} |\mu(\phi)|^{\alpha/2},$$

$$(4.18) \quad \int_E \widetilde{\rho}_{\frac{\alpha}{2},1}(rc_n|S_n(\phi)|^{-1}, \infty) d\mu \rightarrow r^{-\alpha/2} C_{\alpha/2} |\mu(\phi)|^{\alpha/2},$$

and

$$(4.19) \quad \int_E \frac{S_n(\phi)}{c_n} \int_0^{c_n|S_n(\phi)|^{-1}} \widetilde{\rho}_{\frac{\alpha}{2},1}(x, \infty) dx d\mu \rightarrow \frac{2C_{\alpha/2}}{2-\alpha} \text{sgn}(\mu(\phi)) |\mu(\phi)|^{\alpha/2}$$

($\text{sgn}(u) = u/|u|$ if $u \neq 0$ and $\text{sgn}(0) = 0$). We only prove (4.17), because (4.18) and (4.19) can be handled analogously.

For (4.17), we need to use the result in Lemma 6.2

$$\frac{S_n(\phi)}{a_n} \Rightarrow \mu(\phi) \Gamma(1+\beta) M_\beta(1-V_\beta) \quad \text{in } \mathbb{R},$$

where the weak convergence takes place under a probability measure $\mu_n(\cdot) = \mu(\cdot \cap \{\varphi \leq n\})/\mu(\varphi \leq n)$. ($M_\beta(t)$) is the Mittag-Leffler process defined on some probability space $(\Omega', \mathcal{F}', P')$ (the definition is given in (6.1)), and V_β is a random variable defined on the same probability space with density given by (6.3). Here, $M_\beta(t)$ and V_β are independent.

Applying the Skorohod's embedding theorem, there exist random variables Y and Y_n , $n = 1, 2, \dots$ defined on some probability space $(\Omega^*, \mathcal{F}^*, P^*)$ such that

$$P^* \circ Y_n^{-1} = \mu_n \circ \left(\frac{S_n(\phi)}{a_n} \right)^{-1}, \quad n = 1, 2, \dots,$$

$$P^* \circ Y^{-1} = P' \circ (\mu(\phi)\Gamma(1+\beta)M_\beta(1-V_\beta))^{-1},$$

$$Y_n \rightarrow Y \quad P^*\text{-a.s..}$$

Let $\psi(y) = y^{-2} \int_0^y x \tilde{\rho}_{\frac{\alpha}{2},1}(x, \infty) dx$, then we can proceed

$$\begin{aligned} \int_E \left(\frac{S_n(\phi)}{c_n} \right)^2 \int_0^{rc_n|S_n(\phi)|^{-1}} x \tilde{\rho}_{\frac{\alpha}{2},1}(x, \infty) dx d\mu &= \int_E \psi \left(\frac{c_n}{|S_n(\phi)|} \right) d\mu \\ &= \mu(\varphi \leq n) E^* \left[\psi \left(\frac{c_n}{a_n|Y_n|} \right) \right]. \end{aligned}$$

It follows from (4.5) that $c_n a_n^{-1} |Y_n|^{-1} \rightarrow \infty$, P^* -a.s.. Therefore, Karamata's theorem (e.g., Theorem 0.6 in Resnick (1987)) yields

$$\psi \left(\frac{c_n}{a_n|Y_n|} \right) \sim \frac{r^2}{2-\alpha/2} \tilde{\rho}_{\frac{\alpha}{2},1}(rc_n a_n^{-1} |Y_n|^{-1}, \infty) \quad \text{as } n \rightarrow \infty, \quad P^*\text{-a.s.}$$

From uniform convergence theorem of regularly varying functions of negative indices (e.g., Proposition 0.5 in Resnick (1987)), we can say that

$$\tilde{\rho}_{\frac{\alpha}{2},1}(rc_n a_n^{-1} |Y_n|^{-1}, \infty) \sim r^{-\alpha/2} |Y_n|^{\alpha/2} \tilde{\rho}_{\frac{\alpha}{2},1}(c_n a_n^{-1}, \infty) \quad \text{as } n \rightarrow \infty, \quad P^*\text{-a.s.}$$

From (4.4), (4.12), and Proposition 4.2,

$$\begin{aligned} \mu(\varphi \leq n) \psi \left(\frac{c_n}{a_n|Y_n|} \right) &\sim \frac{r^{2-\alpha/2}}{2-\alpha/2} \mu(\varphi \leq n) |Y_n|^{\alpha/2} \tilde{\rho}_{\frac{\alpha}{2},1}(c_n a_n^{-1}, \infty) \\ &\rightarrow \frac{r^{2-\alpha/2} C_{\alpha/2}}{2-\alpha/2} C_{\alpha,\beta}^{-\alpha/2} |Y|^{\alpha/2} \quad \text{as } n \rightarrow \infty, \quad P^*\text{-a.s.} \end{aligned}$$

Integrating the limit yields

$$E^* \left[\frac{r^{2-\alpha/2} C_{\alpha/2}}{2-\alpha/2} C_{\alpha,\beta}^{-\alpha/2} |Y|^{\alpha/2} \right] = \frac{r^{2-\alpha/2} C_{\alpha/2}}{2-\alpha/2} |\mu(\phi)|^{\alpha/2},$$

which is exactly the right hand side of (4.17). Now, to complete the proof, we need to justify taking the limit under the integral. For this, we will apply the so-called Pratt's lemma (see Pratt (1960)). According to Pratt's lemma, we must find a sequence of measurable functions G_0, G_1, \dots defined on $(\Omega^*, \mathcal{F}^*, P^*)$ such that

$$(4.20) \quad \mu(\varphi \leq n) \psi \left(\frac{c_n}{a_n|Y_n|} \right) \leq G_n \quad P^*\text{-a.s.,} \quad n = 1, 2, \dots,$$

$$(4.21) \quad G_n \rightarrow G_0 \quad \text{as } n \rightarrow \infty \quad P^*\text{-a.s., and}$$

$$(4.22) \quad E^* G_n \rightarrow E^* G_0 \quad \text{as } n \rightarrow \infty.$$

For (4.20), there is a $C_1 > 0$ such that

$$\mu(\varphi \leq n) \psi \left(\frac{c_n}{a_n|Y_n|} \right) \leq C_1 \frac{\psi(c_n a_n^{-1} |Y_n|^{-1})}{\psi(c_n a_n^{-1})}$$

because $\mu(\varphi \leq n)\psi(c_n a_n^{-1})$ has a positive and finite limit.

Applying the Potter bound, for any fixed $0 < \xi < \min(\alpha, 2 - \alpha)$, we have

$$\frac{\psi(c_n a_n^{-1} |Y_n|^{-1})}{\psi(c_n a_n^{-1})} \mathbf{1}_{\{c_n > a_n |Y_n|\}} \leq C_2 (|Y_n|^{(\alpha-\xi)/2} + |Y_n|^{(\alpha+\xi)/2})$$

for some $C_2 > 0$.

Since ψ is bounded on $(0, 1]$, for some constant $C_3 \geq C_2$,

$$\frac{\psi(c_n a_n^{-1} |Y_n|^{-1})}{\psi(c_n a_n^{-1})} \mathbf{1}_{\{c_n \leq a_n |Y_n|\}} \leq \frac{C_3}{\psi(c_n a_n^{-1})} \frac{a_n}{c_n} |Y_n|.$$

Therefore, we may write

$$\mu(\varphi \leq n)\psi\left(\frac{c_n}{a_n |Y_n|}\right) \leq C_3 \left(|Y_n|^{(\alpha-\xi)/2} + |Y_n|^{(\alpha+\xi)/2} + \frac{a_n}{c_n} \frac{|Y_n|}{\psi(c_n a_n^{-1})}\right).$$

Now, (4.20) is obtained by taking

$$G_n = C_3 \left(|Y_n|^{(\alpha-\xi)/2} + |Y_n|^{(\alpha+\xi)/2} + \frac{a_n}{c_n} \frac{|Y_n|}{\psi(c_n a_n^{-1})}\right), \quad n = 1, 2, \dots$$

Let

$$G_0 = C_3 \left(|Y|^{(\alpha-\xi)/2} + |Y|^{(\alpha+\xi)/2}\right).$$

We know that $a_n c_n^{-1} \in RV_{-2(1-\beta)/\alpha}$ and $\psi(c_n a_n^{-1}) \in RV_{\beta-1}$; thus,

$$\frac{a_n}{c_n} \frac{1}{\psi(c_n a_n^{-1})} \rightarrow 0$$

from which (4.21) follows.

To show (4.22), recall that $\sup_{n \geq 1} E^* |Y_n| < \infty$ (see the proof of Proposition 4.2). Thus, $(|Y_n|^{(\alpha \pm \xi)/2}, n \geq 1)$ is uniformly integrable with respect to P^* , which in turn implies (4.22). Now, Pratt's lemma is applicable and (4.17) is complete. \square

Proposition 4.5.

$$\frac{1}{c_n} Y_{n,1}''(h) \xrightarrow{p} 0, \quad h = 0, 1, 2, \dots$$

Proof. Choose $\xi > 0$ as specified in Lemma 6.4, 6.5, or 6.6 in accordance with the values of α and β . Let $\alpha' = \alpha - \xi$. For $i \neq j$, we set

$$W_{ij}^{(n, \alpha')} = \frac{1}{c_n} \sum_{k=1}^n f_k(V_i) f_{k+h}(V_j) q(V_i)^{-1/\alpha'} q(V_j)^{-1/\alpha'}.$$

Because of those lemmas, we eventually obtain

$$(4.23) \quad E \left| W_{ij}^{(n, \alpha')} \right|^{\alpha'} \rightarrow 0, \quad \text{for } i \neq j.$$

We will basically follow the argument in Proposition 4.3 of Resnick et al. (1999). Recall that $Y''_{n,1}(h)/c_n$ can be represented by doubly infinite series

$$\frac{1}{c_n} Y''_{n,1}(h) = \sum_{i \neq j} \epsilon_i \epsilon_j U_{\alpha,1}^{\leftarrow} \left(\frac{\Gamma_i q(V_i)}{2} \right) U_{\alpha,1}^{\leftarrow} \left(\frac{\Gamma_j q(V_j)}{2} \right) \frac{1}{c_n} \sum_{k=1}^n f_k(V_i) f_{k+h}(V_j) \equiv \sum_{i \neq j} \widetilde{W}_{ij}^{(n)}.$$

Owing to symmetry of the doubly infinite sum, we only have to consider the case $i < j$, and we will indeed show that $\sum_{i < j} \widetilde{W}_{ij}^{(n)} \xrightarrow{p} 0$. According to Lemma 6.7, there exist an integer m_0 and constants $C > 0$ and $\gamma < \alpha'$ such that for any $m \geq m_0$, all the inequalities given in (a) and (b) of Lemma 6.7 hold.

We then decompose $\sum_{i < j} \widetilde{W}_{ij}^{(n)}$ into three summands

$$\sum_{i < j} \widetilde{W}_{ij}^{(n)} = \sum_{i=1}^{m_0} \sum_{j=i+1}^{m_0} \widetilde{W}_{ij}^{(n)} + \sum_{i=1}^{m_0} \sum_{j=m_0+1}^{\infty} \widetilde{W}_{ij}^{(n)} + \sum_{m_0 < i < j < \infty} \widetilde{W}_{ij}^{(n)}.$$

Now, we only need to prove the following:

$$\begin{aligned} (i) : & \quad \widetilde{W}_{ij}^{(n)} \xrightarrow{p} 0 \quad \text{for all } i, j; \\ (ii) : & \quad \sum_{j=m_0+1}^{\infty} \widetilde{W}_{ij}^{(n)} \xrightarrow{p} 0 \quad \text{for all } i; \\ (iii) : & \quad \sum_{m_0 < i < j < \infty} \widetilde{W}_{ij}^{(n)} \xrightarrow{p} 0. \end{aligned}$$

By the bound $U_{\alpha,1}^{\leftarrow}(x) < Cx^{-1/\alpha'}$ and (4.23), it is evident that $\widetilde{W}_{ij}^{(n)}$ converges to 0 in probability, which proves (i). For (ii) and (iii), by virtue of the inequalities given in Lemma 6.7, it suffices to show that

$$E(|W_{ij}^{(n,\alpha')}|^{\alpha'} (1 + \ln_+^2 |W_{ij}^{(n,\alpha')}|)) \rightarrow 0, \quad i \neq j.$$

To show this, let

$$B^{(n,\alpha')}(x, y) = \frac{1}{c_n} \sum_{k=1}^n f_k(x) f_{k+h}(y) q(x)^{-1/\alpha'} q(y)^{-1/\alpha'}.$$

Then, we can show that

$$\sup_{x, y \in E} |B^{(n,\alpha')}(x, y)| = \mathcal{O}(n^{2(1-\beta)(1/\alpha' - 1/\alpha)}).$$

For the proof, it is important to note that the choice of the density q does not affect the distribution of $Y''_{n,1}(h)$; therefore, we can particularly take

$$q(x) = Q(x) \left(\int_E Q(u) d\mu \right)^{-1},$$

where

$$Q(x) = \max \left(q_0(x), \left(\sum_{k=1}^{n+h} f_k(x)^2 \right)^{\alpha'/2} \right).$$

Here, $q_0 : E \rightarrow (0, \infty)$ is an arbitrarily selected, strictly positive density.

By the Cauchy-Schwarz inequality,

$$\begin{aligned} \sup_{x, y \in E} |B^{(n, \alpha')}(x, y)| &\leq \sup_{x, y \in E} \frac{1}{c_n} \left(\sum_{k=1}^n f_k(x)^2 \right)^{1/2} \left(\sum_{k=1}^n f_{k+h}(y)^2 \right)^{1/2} q(x)^{-1/\alpha'} q(y)^{-1/\alpha'} \\ &\leq \frac{1}{c_n} \left(1 + \int_E \left(\sum_{k=1}^{n+h} f_k(x)^2 \right)^{\alpha'/2} d\mu \right)^{2/\alpha'} \in RV_{2(1-\beta)(\frac{1}{\alpha'} - \frac{1}{\alpha})}, \end{aligned}$$

where the last regular variation index is obtained from (4.5).

Now, we have

$$E(|W_{ij}^{(n, \alpha')}|^{\alpha'} (1 + \ln_+^2 |W_{ij}^{(n, \alpha')}|)) \leq (1 + \ln_+^2 \sup_{x, y \in E} |B^{(n, \alpha')}(x, y)|) E|W_{ij}^{(n, \alpha')}|^{\alpha'}.$$

Observing that $E|W_{ij}^{(n, \alpha')}|^{\alpha'}$ has a negative regular variation index (see the proofs of Lemmas 6.4, 6.5, and 6.6), the right hand side vanishes as $n \rightarrow \infty$. \square

The combination of the next two propositions will prove $(Y'_{n,2}(0) + Y''_{n,2}(0))/c_n \xrightarrow{p} 0$, which can finish the proof of Theorem 4.1.

Proposition 4.6.

$$\frac{1}{c_n} Y'_{n,2}(0) = \frac{1}{c_n} \int_E S_n(f^2)(x) d\widetilde{M}_2(x) \xrightarrow{p} 0.$$

Proof. From the standard argument for convergence in law of the sequence of infinitely divisible random variables (e.g., Theorem 13.14 in Kallenberg (1997)), we only have to check

$$\begin{aligned} \int_E \left(\frac{S_n(f^2)}{c_n} \right)^2 \int_0^{c_n S_n(f^2)^{-1}} x \widetilde{\rho}_{\frac{\alpha}{2}, 2}(x, \infty) dx d\mu &\rightarrow 0, \\ \int_E \widetilde{\rho}_{\frac{\alpha}{2}, 2}(c_n S_n(f^2)^{-1}, \infty) d\mu &\rightarrow 0, \end{aligned}$$

and

$$\int_E \frac{S_n(f^2)}{c_n} \int_0^{c_n S_n(f^2)^{-1}} \widetilde{\rho}_{\frac{\alpha}{2}, 2}(x, \infty) dx d\mu \rightarrow 0.$$

However, an obvious upper bound $\widetilde{\rho}_{\frac{\alpha}{2}, 2}(x, \infty) \leq Cx^{-p_0/2}$, $x > 0$, and the integrability condition $f \in L^2(\mu)$ easily prove these limits. \square

Proposition 4.7.

$$\frac{1}{c_n} Y''_{n,2}(0) = \sum_{i \neq j} \epsilon_i \epsilon_j U_{\alpha, 2}^{\leftarrow} \left(\frac{\Gamma_i q(V_i)}{2} \right) U_{\alpha, 2}^{\leftarrow} \left(\frac{\Gamma_j q(V_j)}{2} \right) \frac{1}{c_n} \sum_{k=1}^n f_k(V_i) f_k(V_j) \xrightarrow{p} 0.$$

Proof. The proof is analogous to that of Proposition 4.5. Taking advantage of the inequalities given in Lemma 6.7 (see also Remark 6.8), the proof will be finished if

$$E(|W_{ij}^{(n, p_0)}|^{p_0} (1 + \ln_+^2 |W_{ij}^{(n, p_0)}|)) \rightarrow 0, \quad i \neq j.$$

The argument for showing this is mostly the same as in Proposition 4.5, and so we omit it. \square

5. EXAMPLES

We present three examples of different situations where Theorem 4.1 applies. The first example is what Resnick et al. (2000) studied, but their example can be regarded as a special case of our more general setup.

Example 5.1. Let $(x_k, k \geq 0)$ be an irreducible null recurrent Markov chain with state space \mathbb{Z} and transition matrix $P = (p_{ij})$. Let $P_i(\cdot)$ be a probability law of (x_k) starting in state $i \in \mathbb{Z}$. Since (x_k) is null recurrent, there exists a unique (up to constant multiplications), infinite, invariant measure (π_i) . We set $\pi_0 = 1$ for normalization. Define a σ -finite and infinite measure on $(E, \mathcal{E}) = (\mathbb{Z}^{\mathbb{N}}, \mathcal{B}(\mathbb{Z}^{\mathbb{N}}))$ by

$$\mu(\cdot) = \sum_{i \in \mathbb{Z}} \pi_i P_i(\cdot).$$

Let $T : \mathbb{Z}^{\mathbb{N}} \rightarrow \mathbb{Z}^{\mathbb{N}}$ be the left shift map defined by $T(x_0, x_1, \dots) = (x_1, x_2, \dots)$. Obviously, T preserves the measure μ . From Harris and Robbins (1953), it is known that the map T is conservative and ergodic.

We consider the set $A = \{x \in \mathbb{Z}^{\mathbb{N}} : x_0 = 0\}$. From the result in Section 4.5 of Aaronson (1997), we have

$$\hat{T}^k \mathbf{1}_A(x) = P_0(x_k = 0) \quad \text{for } x \in A.$$

Thus, with the normalizing sequence $a_n = \sum_{k=1}^n P_0(x_k = 0)$,

$$\frac{1}{a_n} \sum_{k=1}^n \hat{T}^k \mathbf{1}_A(x) = 1 = \mu(A)$$

holds for every $x \in A$. Here, A is a Darling-Kac set and hence T is a pointwise dual ergodic map.

One of the possible ways for ensuring regular variation of (a_n) is to assume

$$\sum_{k=1}^n P_0(\varphi \geq k) \in RV_{1-\beta} \quad \text{for some } 0 \leq \beta < 1,$$

where $\varphi(x) = \min\{n \geq 1 : x_n = 0\}$, $x \in \mathbb{Z}^{\mathbb{N}}$, is the first entrance time to the set A .

From Lemma 3.3 in Resnick et al. (2000), we see that $\mu(\varphi = n) = P_0(\varphi \geq n)$ and by (3.7),

$$a_n \sim \frac{1}{\Gamma(2-\beta)\Gamma(1+\beta)} \frac{n}{\mu(\varphi \leq n)} \in RV_{\beta}.$$

We will proceed to check condition (4.2). The formula on p. 156 of Aaronson (1997)) gives

$$\hat{T}^k \mathbf{1}_{A^c \cap \{\varphi=k\}}(x_0, x_1, \dots) = \mathbf{1}_{\{x_0=0\}} \sum_{i_0 \neq 0} \pi_{i_0} \sum_{i_1 \neq 0} p_{i_0 i_1} \cdots \sum_{i_{k-1} \neq 0} p_{i_{k-2} i_{k-1}} p_{i_{k-1} 0},$$

which immediately implies (4.2).

We take a measurable function $f : \mathbb{Z}^{\mathbb{N}} \rightarrow \mathbb{R}$ that is supported by the set A and satisfies (2.2). Now, Theorem 4.1 applies if the parameters lie in the range $1 < \alpha < 2$, $0 \leq \beta < 1$, or $0 < \alpha \leq 1$, $0 \leq \beta < 1/(2 - \alpha)$.

On the other hand, if $0 < \alpha \leq 1$ and $1/(2 - \alpha) \leq \beta < 1$, we need to check the conditions given in (i) of Theorem 4.1. For this, we consider a two-dimensional Markov chain $((x_k, y_k), k \geq 0)$ with (y_k) an independent copy of (x_k) . Let $P_{(i,j)}(\cdot)$ be a probability law of (x_k, y_k) starting from $(i, j) \in \mathbb{Z} \times \mathbb{Z}$. It is now easy to check that (x_k, y_k) is also irreducible and null recurrent, and a probability measure $\mu \times \mu$ can be written as

$$(\mu \times \mu)(\cdot) = \sum_{i \in \mathbb{Z}} \sum_{j \in \mathbb{Z}} \pi_i \pi_j P_{(i,j)}(\cdot).$$

Because of Harris and Robbins (1953) again, we can say that $T \times T$ is conservative ergodic and measure preserving map on $(\mathbb{Z}^{\mathbb{N}} \times \mathbb{Z}^{\mathbb{N}}, \mathcal{B}(\mathbb{Z}^{\mathbb{N}}) \times \mathcal{B}(\mathbb{Z}^{\mathbb{N}}))$.

Evidently, the product set $A \times A$ is a Darling-Kac set. Indeed,

$$\sum_{k=1}^n (\widehat{T \times T})^k \mathbf{1}_{A \times A}(x, y) = \sum_{k=1}^n \widehat{T}^k \mathbf{1}_A(x) \widehat{T}^k \mathbf{1}_A(y) = \sum_{k=1}^n P_0(x_k = 0)^2 \quad \text{for } (x, y) \in A \times A.$$

Therefore, by the normalizing sequence $a'_n = \sum_{k=1}^n P_0(x_k = 0)^2$, the product set $A \times A$ turns out to be a Darling-Kac set, and $T \times T$ is of course pointwise dual ergodic.

Once again, by appealing to Lemma 3.3 in Resnick et al. (2000), we get

$$(\mu \times \mu)(\varphi(x, y) \leq n) = \sum_{k=1}^n P_{(0,0)}(\varphi(x, y) \geq k) \in RV_{2(1-\beta)}.$$

Thus,

$$a'_n \sim \frac{1}{\Gamma(3 - 2\beta)\Gamma(2\beta)} \frac{n}{(\mu \times \mu)(\varphi(x, y) \leq n)} \in RV_{2\beta-1}.$$

To check (4.10), one more application of the formula on p. 156 of Aaronson (1997) yields

$$\begin{aligned} & (\widehat{T \times T})^k \mathbf{1}_{(A \times A)^c \cap \{\varphi(x, y) = k\}}((x_0, y_0), (x_1, y_1) \dots) \\ &= \mathbf{1}_{\{(x_0, y_0) = (0, 0)\}} \sum_{(i_0, j_0) \neq (0, 0)} \pi_{i_0} \pi_{j_0} \sum_{(i_1, j_1) \neq (0, 0)} p_{i_0 i_1} p_{j_0 j_1} \dots \sum_{(i_{k-1}, j_{k-1}) \neq (0, 0)} p_{i_{k-2} i_{k-1}} p_{i_{k-1} 0} p_{j_{k-2} j_{k-1}} p_{j_{k-1} 0}. \end{aligned}$$

Therefore (4.10) holds, and in this case, Theorem 4.1 applies as well.

It is not difficult to prove that the process $\mathbf{X} = (X_1, X_2, \dots)$ is mixing. To see this, we only check a sufficient condition proposed by Theorem 5 in Rosiński and Żak (1996):

$$\mu\{x : |f(x)| > \epsilon, |f \circ T^n(x)| > \epsilon\} \rightarrow 0 \quad \text{as } n \rightarrow \infty, \quad \text{for every } \epsilon > 0.$$

Since f vanishes outside of A and (x_n) is null recurrent, as $n \rightarrow \infty$, we have

$$\mu\{x : |f(x)| > \epsilon, |f \circ T^n(x)| > \epsilon\} \leq \mu(A \cap T^{-n}A) = P_0(x_n = 0) \rightarrow 0.$$

The next two examples are less familiar to probabilists, but are well known to ergodic theorists.

Example 5.2. In this example, we will define the so-called *basic AFN-system*. We refer the reader to Zweimüller (2000) and to Thaler and Zweimüller (2006). Let E be the union of a finite family of disjoint bounded open intervals on \mathbb{R} , and let \mathcal{E} be its Borel σ -field. Let ξ be a (possibly infinite) collection of nonempty, pairwise disjoint open subintervals in E . With λ being the one-dimensional Lebesgue measure, we assume $\lambda(E \setminus \bigcup_{Z \in \xi} Z) = 0$.

Let $T : E \rightarrow E$ be a twice-differentiable map and strictly monotonic on each $Z \in \xi$. Suppose that T satisfies the following conditions.

(A) *Adler's condition:*

$$T''/(T')^2 \text{ is bounded on } \bigcup_{Z \in \xi} Z.$$

(F) *Finite image condition:*

$$\text{the collection } T\xi = \{TZ : Z \in \xi\} \text{ is finite.}$$

(N) *A possibility of nonuniform expansion:* There exists a finite subset $\zeta \subseteq \xi$ such that each $Z \in \zeta$ has an *indifferent fixed point* x_Z as one of its endpoints. That is,

$$\lim_{x \rightarrow x_Z, x \in Z} Tx = x_Z \quad \text{and} \quad \lim_{x \rightarrow x_Z, x \in Z} T'x = 1.$$

Moreover, we suppose that for each $Z \in \zeta$,

$$\text{either } T' \text{ decreases on } (-\infty, x_Z) \cap Z \text{ or increases on } (x_Z, \infty) \cap Z,$$

depending on whether x_Z is the left endpoint or the right endpoint of Z .

Assume that T is uniformly expanding whenever it is bounded away from the family of indifferent fixed points $\{x_Z : Z \in \zeta\}$, i.e., for each $\epsilon > 0$, there is a $\rho(\epsilon) > 1$ such that

$$|T'| \geq \rho(\epsilon) \text{ on } E \setminus \bigcup_{Z \in \zeta} ((x_Z - \epsilon, x_Z + \epsilon) \cap Z).$$

We will further specify the behavior of T in neighborhood of the indifferent fixed points; for every $Z \in \zeta$, there is $0 \leq \beta_Z < 1$ such that

$$(5.1) \quad Tx = x + a_Z |x - x_Z|^{1/\beta_Z + 1} + o(|x - x_Z|^{1/\beta_Z + 1}) \quad \text{as } x \rightarrow x_Z \text{ in } Z$$

for some $a_Z \neq 0$.

As argued in Zweimüller (2000), there is not much loss of generality in assuming that T is conservative and ergodic with respect to λ . In this case, if additionally ζ is non-empty, then the triplet (E, T, ξ) is said to be a *basic AFN-system*. In the sequel, we will assume this property.

Given a basic AFN-system (E, T, ξ) , there always exists an infinite invariant measure $\mu \ll \lambda$ with density $d\mu/d\lambda(x) = h_0(x)G(x)$, where

$$G(x) = \begin{cases} (x - x_Z)(x - (T|_Z)^{-1}(x))^{-1} & \text{if } x \in Z \in \zeta, \\ 1 & \text{if } x \in E \setminus \bigcup_{Z \in \zeta} Z, \end{cases}$$

and h_0 is a function of bounded variation, bounded away from both zero and infinity. Now, we can view T as a conservative ergodic and measure preserving map on an infinite measure space (E, \mathcal{E}, μ) .

An example of a basic AFN-map is Boole's transformation placed on $E = (0, 1/2) \cup (1/2, 1)$, defined by

$$T(x) = \frac{x(1-x)}{1-x-x^2}, \quad x \in (0, 1/2), \quad T(x) = 1 - T(1-x), \quad x \in (1/2, 1).$$

It admits expansions of the form (5.1) at the indifferent fixed points $x_Z = 0$ and $x_Z = 1$ with $\beta_Z = 1/2$ in both cases. The invariant measure μ satisfies

$$\frac{d\mu}{d\lambda}(x) = \frac{1}{x^2} + \frac{1}{(1-x)^2}, \quad x \in E.$$

See Thaler (2001).

Given a constant $0 < \epsilon < 1$, we take

$$A = E \setminus \bigcup_{Z \in \zeta} ((x_Z - \epsilon, x_Z + \epsilon) \cap Z).$$

Since $\lambda(\partial A) = 0$ and A is bounded away from $\{x_Z, Z \in \zeta\}$, A is a Darling-Kac set, and hence T is a pointwise dual ergodic map (Corollary 3 in Zweimüller (2000)). Furthermore, because of the assumption (5.1), (a_n) turns out to be regularly varying with index $\beta = \min_{Z \in \zeta} \beta_Z$ (Theorem 4 in Zweimüller (2000)). Moreover, the formulas (2.5) and (2.6) in Thaler and Zweimüller (2006) prove the condition (4.2).

Suppose that the parameters α and β lie in the range of either $1 < \alpha < 2$, $0 \leq \beta < 1$, or $0 < \alpha \leq 1$, $0 \leq \beta < 1/(2 - \alpha)$. If a measurable function $f : E \rightarrow \mathbb{R}$ is supported by the set A together with a proper integrability assumption, then Theorem 4.1 applies.

Suppose that $0 < \alpha \leq 1$ and $1/(2 - \alpha) \leq \beta < 1$. In this case, we will check (ii) in Theorem 4.1, because unlike Example 5.1, the product map $T \times T$ is not generally conservative and ergodic. According to condition (ii), however, the Darling-Kac set A must be a uniformly returning set. Unfortunately, this is not always the case for a general basic AFN-system. To overcome this difficulty, we have to impose certain additional assumptions; see for example, Thaler (2000). If we restrict ourselves to such a type of a basic AFN-system, then (ii) is satisfied and consequently Theorem 4.1 follows.

Finally, it is worth pointing out that the process $\mathbf{X} = (X_1, X_2, \dots)$ is mixing. This can be proved as in Example 5.1.

Example 5.3. We will construct the dynamical system by a S -unimodal map with flat critical point. The main reference here is Zweimüller (2004). Let $T : [a, b] \rightarrow [a, b]$ be a S -unimodal map with flat critical point $c \in (a, b)$. That is, the Schwarzian derivative of T is nonpositive: $ST = T'''/T' - \frac{3}{2}(T''/T')^2 \leq 0$, and all derivatives at the critical point c vanish: $T^{(n)}c = 0$ for all $n \geq 1$. Further assume that $Ta = Tb = a$ and that $\int_{[a,b]} \ln |T'| d\lambda = -\infty$ (λ is the one-dimensional Lebesgue measure). In addition, we suppose that T satisfies Misiurewicz condition, i.e., there is an open interval I containing c such that $T^n c \notin I$ for all $n \geq 1$. Also, assume that there exists a positive and finite Lyapunov exponent $\lambda_c = \lim_{n \rightarrow \infty} n^{-1} \ln |(T^n)'(Tc)|$.

The dynamical effect of a flat critical point is that the closer the orbit gets to c , the slower it moves away from the critical orbit ($T^n c, n \geq 1$). Consequently, the orbit stays in neighborhood of ($T^n c, n \geq 1$) for a nonnegligible amount of time.

It is shown in Zweimüller (2004) that there exists an infinite measure $\mu \ll \lambda$ such that T is a conservative ergodic and measure preserving map on $([a, b], \mathcal{B}([a, b]), \mu)$.

From Zweimüller (2004) and Zweimüller (2007), one can find a Darling-Kac set A , which is bounded away from the critical orbit ($T^n c, n \geq 1$) such that

$$\left(\frac{\hat{T}^n \mathbf{1}_{A \cap \{\varphi=n\}}}{\mu(A \cap \{\varphi=n\})}, n \geq 1 \right) \text{ is bounded on } A.$$

This property in fact proves condition (4.2). The existence of a positive and finite Lyapunov exponent guarantees that the normalizing sequence (a_n) for the Darling-Kac set is a regularly varying function of the order $0 < \beta < 1$ (Theorem 7 in Zweimüller (2004)).

Suppose that the range of the parameters α and β is either $1 < \alpha < 2$, $0 < \beta < 1$ or $0 < \alpha \leq 1$, $0 < \beta < 1/(2 - \alpha)$. If a measurable function f satisfies a proper integrability condition and is supported by the set A , Theorem 4.1 applies.

6. APPENDIX

For $0 < \beta < 1$, let $(S_\beta(t), t \geq 0)$ be a β -stable subordinator, i.e., a stable Lévy process with increasing sample path. Assume that the moment generating function of $(S_\beta(t))$ is given by $E \exp\{-\theta S_\beta(t)\} = \exp\{-t\theta^\beta\}$ for $\theta > 0$ and $t \geq 0$. Define its inverse process by

$$(6.1) \quad M_\beta(t) = S_\beta^-(t) = \inf\{u \geq 0 : S_\beta(u) \geq t\}, \quad t \geq 0.$$

We call the process $(M_\beta(t))$ the Mittag-Leffler process with index β because the moment generating function of $(M_\beta(t))$ is given by the Mittag-Leffler function

$$(6.2) \quad E \exp\{\theta M_\beta(t)\} = \sum_{n=0}^{\infty} \frac{(\theta t^\beta)^n}{\Gamma(1 + n\beta)}, \quad \theta \in \mathbb{R};$$

see Proposition 1(a) in Bingham (1971).

The Mittag-Leffler process is well-defined in the limiting case $\beta = 0$ as well. From the expression (6.2), it is natural to regard $M_0(t) \equiv M_0$ as an exponential random variable of unit parameter.

For $0 \leq \beta < 1$, let V_β denote a random variable with density

$$(6.3) \quad g_{V_\beta}(x) = (1 - \beta)x^{-\beta}, \quad 0 < x \leq 1.$$

Here, V_β is taken to be independent of $(M_\beta(t))$.

Remark 6.1. A formal substitution of $\beta = 1$ into (6.2) indicates that $M_1(t)$ should be regarded as a straight line process, i.e., $M_1(t) = t$, $t \geq 0$. A straight line process can be viewed as the inverse of the degenerate 1-stable subordinator $S_1(t) = t$, $t \geq 0$. If one interprets $V_1 = 0$, we possibly establish not only Lemma 6.2 discussed below but also our main theorem. However, the proof is very sensitive to the behavior of slowly varying functions that appear in the local Lévy measure $\rho_\alpha(x, \infty)$ and the normalizing sequence (a_n) . Thus, regarding the limiting cases, we restrict ourselves to the case $\beta = 0$ and omit the details for $\beta = 1$.

Lemma 6.2. *Under the assumptions of Theorem 4.1, let $\phi(x) = f(x) \sum_{h=0}^H \theta_h f_h(x)$, $\theta_0, \dots, \theta_H \in \mathbb{R}$. Then, we have*

$$\frac{S_n(\phi)}{a_n} \Rightarrow \mu(\phi)\Gamma(1 + \beta)M_\beta(1 - V_\beta) \quad \text{in } \mathbb{R}$$

with respect to $\mu_n(\cdot) = \mu(\cdot \cap \{\varphi \leq n\})/\mu(\varphi \leq n)$.

Proof. We first claim that

$$(6.4) \quad \frac{S_n(\mathbf{1}_A)}{a_n} \Rightarrow \mu(A)\Gamma(1 + \beta)M_\beta(1 - V_\beta) \quad \text{in } \mathbb{R}$$

with respect to μ_n , and try to replace $\mathbf{1}_A$ by a more general function ϕ thereafter.

Because of (6.2) and the fact that $M_\beta(t)$ is a self-similar process with self-similarity exponent β , the moments of $M_\beta(1 - V_\beta)$ are given by

$$EM_\beta(1 - V_\beta)^r = E(1 - V_\beta)^{r\beta}EM_\beta(1)^r = r! \frac{\Gamma(2 - \beta)}{\Gamma(r\beta + 2 - \beta)}.$$

Recall the fact that given the moments of any order, the Mittag-Leffler laws can be uniquely determined (e.g., Bingham (1971)). A simple application of the Carleman sufficient condition

proves that the laws of $M_\beta(1 - V_\beta)$ can also be uniquely determined by their moments. From these observations, (6.4) follows if we can show that

$$\int_E \left(\frac{S_n(\mathbf{1}_A)}{a_n} \right)^r d\mu_n \rightarrow (\mu(A)\Gamma(1 + \beta))^r r! \frac{\Gamma(2 - \beta)}{\Gamma(r\beta + 2 - \beta)}, \quad \text{for every } r = 1, 2, \dots$$

From now, we will repeatedly use Karamata's Tauberian theorem for power series (e.g., Corollary 1.7.3 in Bingham et al. (1987)).

First, we claim that

$$(6.5) \quad \sum_{n=1}^{\infty} \left(\int_E \binom{S_n(\mathbf{1}_A)}{r} d\mu \right) e^{-\lambda n} \sim \frac{1}{(r-1)!} \frac{\mu(A)}{\lambda} \sum_{n=1}^{\infty} \left(\int_A S_n(\mathbf{1}_A)^{r-1} d\mu_A \right) e^{-\lambda n} \quad \text{as } \lambda \downarrow 0,$$

where $\mu_A(\cdot) = \mu(\cdot \cap A)/\mu(A)$.

For the proof, the following identity is needed:

$$\binom{S_n(\mathbf{1}_A)}{r} = \sum_{k=1}^n \left(\mathbf{1}_A \binom{S_{n-k}(\mathbf{1}_A)}{r-1} \right) \circ T^k, \quad r = 1, 2, \dots$$

As $\lambda \downarrow 0$, we have

$$\begin{aligned} \sum_{n=1}^{\infty} \left(\int_E \binom{S_n(\mathbf{1}_A)}{r} d\mu \right) e^{-\lambda n} &= \sum_{n=1}^{\infty} \sum_{k=1}^n \left(\int_E \left(\mathbf{1}_A \binom{S_{n-k}(\mathbf{1}_A)}{r-1} \right) \circ T^k d\mu \right) e^{-\lambda n} \\ &\sim \frac{\mu(A)}{\lambda} \sum_{n=1}^{\infty} \left(\int_A \binom{S_n(\mathbf{1}_A)}{r-1} d\mu_A \right) e^{-\lambda n}. \end{aligned}$$

It is elementary to show that

$$\int_A \binom{S_n(\mathbf{1}_A)}{r-1} d\mu_A \sim \frac{1}{(r-1)!} \int_A S_n(\mathbf{1}_A)^{r-1} d\mu_A \quad n \rightarrow \infty,$$

which completes (6.5).

We already know from the proof of Theorem 9.1 in Thaler and Zweimüller (2006) (or Aaronson (1981)) that

$$\begin{aligned} \int_A S_n(\mathbf{1}_A)^{r-1} d\mu_A &\sim (\mu(A)\Gamma(1 + \beta))^{r-1} E M_\beta(1)^{r-1} a_n^{r-1} \\ &= (\mu(A)\Gamma(1 + \beta))^{r-1} (r-1)! \frac{a_n^{r-1}}{\Gamma((r-1)\beta + 1)} \quad \text{as } n \rightarrow \infty. \end{aligned}$$

Since (a_n) is regularly varying with exponent β , one can set $a_n = n^\beta L(n)$ by some slowly varying function L . Then, from Karamata's Tauberian theorem,

$$(6.6) \quad \sum_{n=1}^{\infty} \left(\int_A S_n(\mathbf{1}_A)^{r-1} d\mu_A \right) e^{-\lambda n} \sim (r-1)! (\mu(A)\Gamma(1 + \beta))^{r-1} \frac{1}{\lambda^{(r-1)\beta+1}} L(\lambda^{-1})^{r-1} \quad \text{as } \lambda \downarrow 0.$$

Consequently, from (6.5) and (6.6),

$$\sum_{n=1}^{\infty} \left(\int_E \binom{S_n(\mathbf{1}_A)}{r} d\mu \right) e^{-\lambda n} \sim \mu(A)^r \Gamma(1 + \beta)^{r-1} \frac{1}{\lambda^{r\beta+2-\beta}} L(\lambda^{-1})^{r-1} \quad \text{as } \lambda \downarrow 0.$$

Since $\int_E \binom{S_n(\mathbf{1}_A)}{r} d\mu$ is nondecreasing in n and $r\beta + 2 - \beta > 0$, one more application of Karamata's Tauberian theorem yields

$$\int_E \binom{S_n(\mathbf{1}_A)}{r} d\mu \sim \frac{\mu(A)^r \Gamma(1 + \beta)^{r-1}}{\Gamma(r\beta + 2 - \beta)} n a_n^{r-1} \quad \text{as } n \rightarrow \infty.$$

It is not difficult to justify

$$\int_E \binom{S_n(\mathbf{1}_A)}{r} d\mu \sim \frac{1}{r!} \int_E S_n(\mathbf{1}_A)^r d\mu.$$

Therefore, we get

$$(6.7) \quad \int_E \left(\frac{S_n(\mathbf{1}_A)}{a_n} \right)^r d\mu \sim \mu(A)^r r! \frac{\Gamma(1 + \beta)^{r-1}}{\Gamma(r\beta + 2 - \beta)} \frac{n}{a_n} \quad \text{as } n \rightarrow \infty.$$

Thus we get, from (3.6) and (3.7),

$$\begin{aligned} \int_E \left(\frac{S_n(\mathbf{1}_A)}{a_n} \right)^r d\mu_n &= \frac{1}{\mu(\varphi \leq n)} \int_E \left(\frac{S_n(\mathbf{1}_A)}{a_n} \right)^r d\mu \\ &\rightarrow \mu(A)^r r! \frac{\Gamma(2 - \beta) \Gamma(1 + \beta)^r}{\Gamma(r\beta + 2 - \beta)} \quad \text{as } n \rightarrow \infty, \end{aligned}$$

which completes (6.4).

Next, the indicator function $\mathbf{1}_A$ must be replaced by ϕ . To this end, it suffices to show that

$$(6.8) \quad \mu_n \left(\left| \frac{S_n(\phi)}{S_n(\mathbf{1}_A)} - \frac{\mu(\phi)}{\mu(A)} \right| > \epsilon \right) \rightarrow 0 \quad \text{for every } \epsilon > 0.$$

Indeed, if the above is true, the Slutsky theorem gives

$$\left(\frac{S_n(\mathbf{1}_A)}{a_n}, \frac{S_n(\phi)}{S_n(\mathbf{1}_A)} \right) \Rightarrow \left(\mu(A) \Gamma(1 + \beta) M_\beta (1 - V_\beta), \frac{\mu(\phi)}{\mu(A)} \right)$$

with respect to μ_n . Applying the continuous mapping theorem, we get

$$\frac{S_n(\phi)}{a_n} \Rightarrow \mu(\phi) \Gamma(1 + \beta) M_\beta (1 - V_\beta) \quad \text{in } \mathbb{R}.$$

Since $\mu(A) < \infty$, it is enough to verify

$$\mu_n \left(A^c \cap \left\{ \left| \frac{S_n(\phi)}{S_n(\mathbf{1}_A)} - \frac{\mu(\phi)}{\mu(A)} \right| > \epsilon \right\} \right) \rightarrow 0 \quad \text{for every } \epsilon > 0.$$

Denote

$$K_n = \left\{ \left| \frac{\phi + S_n(\phi)}{1 + S_n(\mathbf{1}_A)} - \frac{\mu(\phi)}{\mu(A)} \right| > \epsilon \right\}.$$

Noting that ϕ is supported by A , we obtain

$$\mu \left(A^c \cap \{\varphi \leq n\} \cap \left\{ \left| \frac{S_n(\phi)}{S_n(\mathbf{1}_A)} - \frac{\mu(\phi)}{\mu(A)} \right| > \epsilon \right\} \right) = \sum_{m=1}^n \mu(A^c \cap \{\varphi = m\} \cap T^{-m} K_{n-m}).$$

Thus, for an arbitrary constant $\delta \in (0, 1)$, one can proceed as follows.

$$\mu_n \left(A^c \cap \left\{ \left| \frac{S_n(\phi)}{S_n(\mathbf{1}_A)} - \frac{\mu(\phi)}{\mu(A)} \right| > \epsilon \right\} \right)$$

$$\begin{aligned}
&\leq \frac{1}{\mu(\varphi \leq n)} \sum_{m=1}^{\lceil (1-\delta)n \rceil} \mu(A^c \cap \{\varphi = m\} \cap T^{-m}K_{n-m}) + \frac{1}{\mu(\varphi \leq n)} \sum_{m=\lceil (1-\delta)n \rceil+1}^n \mu(\varphi = m) \\
&= \int_A \frac{1}{\mu(\varphi \leq n)} \sum_{m=1}^{\lceil (1-\delta)n \rceil} \widehat{T}^m \mathbf{1}_{A^c \cap \{\varphi=m\}} \cdot \mathbf{1}_{K_{n-m}} d\mu + 1 - \frac{\mu(\varphi \leq \lceil (1-\delta)n \rceil)}{\mu(\varphi \leq n)} \\
&\leq \int_A \frac{1}{\mu(\varphi \leq \lceil (1-\delta)n \rceil)} \sum_{m=1}^{\lceil (1-\delta)n \rceil} \widehat{T}^m \mathbf{1}_{A^c \cap \{\varphi=m\}} \sup_{n-\lceil (1-\delta)n \rceil \leq i \leq n} \mathbf{1}_{K_i} d\mu + 1 - \frac{\mu(\varphi \leq \lceil (1-\delta)n \rceil)}{\mu(\varphi \leq n)}.
\end{aligned}$$

Because of (4.2), $\mu(\varphi \leq \lceil (1-\delta)n \rceil)^{-1} \sum_{m=1}^{\lceil (1-\delta)n \rceil} \widehat{T}^m \mathbf{1}_{A^c \cap \{\varphi=m\}}$ is uniformly bounded on A ; further, the Hopf's ergodic theorem (sometimes also called a ratio ergodic theorem; see Theorem 2.2.5 in Aaronson (1997)) yields

$$\sup_{n-\lceil (1-\delta)n \rceil \leq i \leq n} \mathbf{1}_{K_i} \rightarrow 0 \quad \text{as } n \rightarrow \infty \quad \text{a.e. on } A.$$

Applying the dominated convergence theorem, we conclude

$$\limsup_{n \rightarrow \infty} \mu_n \left(A^c \cap \left\{ \left| \frac{S_n(\phi)}{S_n(\mathbf{1}_A)} - \frac{\mu(\phi)}{\mu(A)} \right| > \epsilon \right\} \right) \leq 1 - (1-\delta)^{1-\beta}.$$

Letting $\delta \downarrow 0$ on the right hand side, we get (6.8). \square

Remark 6.3. In Lemma 6.2, we assumed that T is conservative, ergodic, measure preserving, and pointwise dual ergodic with return sequence (a_n) . However, a careful inspection of Theorem 9.1 in Thaler and Zweimüller (2006) reveals the following. Suppose that a measurable map T defined on (E, \mathcal{E}, μ) is measure preserving and satisfies

$$\frac{1}{a_n} \sum_{k=1}^n \widehat{T}^k \mathbf{1}_A \rightarrow \mu(A) \quad \text{uniformly, a.e. on } A$$

(this condition does make sense because \widehat{T} is well-defined as long as T is measure preserving). We assume neither conservativity nor ergodicity for the operator T . However, relation (6.7) still follows. This observation plays an important role in the proof of Lemma 6.6.

Lemmas 6.4, 6.5, and 6.6 below are necessary components for the proof of Propositions 4.5 and 4.7. The most important result established in these lemmas is

$$(6.9) \quad E \left| \frac{1}{c_n} \sum_{k=1}^n f_k(V_i) f_{k+h}(V_j) q(V_i)^{-1/\alpha'} q(V_j)^{-1/\alpha'} \right|^{\alpha'} \rightarrow 0, \quad \text{for } i \neq j.$$

The first lemma treats the case when α and β lie in the range of (4.6).

Lemma 6.4. *Suppose the conditions of Theorem 4.1, where α and β lie in the range of (4.6). We fix a constant $\xi > 0$ such that*

$$\xi < \alpha - 1 \quad \text{if } 1 < \alpha < 2,$$

$$\xi < \alpha \left(1 - \frac{1}{2 - \beta(2 - \alpha)} \right) \quad \text{if } 0 < \alpha \leq 1, \ 0 \leq \beta < \frac{1}{2 - \alpha}.$$

Let $\alpha' = \alpha - \xi$. Then, (6.9) holds.

Proof. First, suppose that $1 < \alpha < 2$. Since $\alpha' > 1$, Minkowski's inequality applies to obtain

$$\begin{aligned} E \left| \frac{1}{c_n} \sum_{k=1}^n f_k(V_i) f_{k+h}(V_j) q(V_i)^{-1/\alpha'} q(V_j)^{-1/\alpha'} \right|^{\alpha'} &= \frac{1}{c_n^{\alpha'}} \int_{E \times E} \left| \sum_{k=1}^n f_k(x) f_k(y) \right|^{\alpha'} (\mu \times \mu)(dx \, dy) \\ &\leq \left(\frac{n}{c_n} \right)^{\alpha'} \left(\int_E |f|^{\alpha'} d\mu \right)^2. \end{aligned}$$

Since $n/c_n \in RV_{(1-\beta)(1-2/\alpha)}$ with $(1-\beta)(1-2/\alpha) < 0$, we have $n/c_n \rightarrow 0$.

Next, suppose that $0 < \alpha \leq 1$ and $0 \leq \beta < 1/(2 - \alpha)$. In this case, a simple application of the triangle inequality gives

$$\begin{aligned} E \left| \frac{1}{c_n} \sum_{k=1}^n f_k(V_i) f_{k+h}(V_j) q(V_i)^{-1/\alpha'} q(V_j)^{-1/\alpha'} \right|^{\alpha'} &= \frac{1}{c_n^{\alpha'}} \int_{E \times E} \left| \sum_{k=1}^n f_k(x) f_k(y) \right|^{\alpha'} (\mu \times \mu)(dx \, dy) \\ &\leq \frac{n}{c_n^{\alpha'}} \left(\int_E |f|^{\alpha'} d\mu \right)^2. \end{aligned}$$

However, we see that $n/c_n^{\alpha'} \in RV_{1-\alpha'(\beta+2(1-\beta)/\alpha)}$ with $1 - \alpha'(\beta + 2(1 - \beta)/\alpha) < 0$. \square

As compared with the case where α and β lie in the range (4.6), establishing (6.9) in the case $0 < \alpha \leq 1$ and $1/(2 - \alpha) \leq \beta < 1$ is more difficult. Indeed, if $0 < \alpha \leq 1$ and $1/(2 - \alpha) \leq \beta < 1$, a simple manipulation of the basic inequalities as we have seen in Lemma 6.4 cannot lead us to (6.9). Thus, we need some alternative approaches.

A possible alternative approach is to assume that the product map $T \times T$ still has *nice properties* as given in (i) of Theorem 4.1. Specifically, in the next lemma, we will assume that $T \times T$ is still conservative and ergodic on a measure space $(E \times E, \mathcal{E} \times \mathcal{E}, \mu \times \mu)$, and further, it is also pointwise dual ergodic. The benefit of this assumption is that we can explicitly calculate the exact growth rate of the quantity $\int_{E \times E} \left| \sum_{k=1}^n f_k(x) f_k(y) \right|^{\alpha'} (\mu \times \mu)(dx \, dy)$.

Lemma 6.5. *Suppose the conditions of Theorem 4.1, where $0 < \alpha \leq 1$ and $1/(2 - \alpha) \leq \beta < 1$, and particularly assume condition (i). We fix $0 < \xi < \alpha^2/(\alpha + 2)$ and let $\alpha' = \alpha - \xi$. Then, (6.9) follows.*

Proof. Denote by $S_n(f \times f)(x, y) = \sum_{k=1}^n f_k(x) f_k(y)$ a partial sum defined on a product space $E \times E$. By virtue of (4.10), proceeding as in the proof of Lemma 6.2, we can get

$$(6.10) \quad \frac{S_n(f \times f)(x, y)}{a'_n} \Rightarrow \mu(f)^2 \Gamma(2\beta) M_{2\beta-1} (1 - V_{2\beta-1}) \quad \text{in } \mathbb{R},$$

where the weak convergence takes place under a probability measure

$$(6.11) \quad (\mu \times \mu)_n(\cdot) = (\mu \times \mu)(\cdot \cap \{\varphi(x, y) \leq n\}) / (\mu \times \mu)(\varphi(x, y) \leq n).$$

Here, $M_{2\beta-1}(t)$ is the Mittag-Leffler process with exponent $2\beta - 1$, and $V_{2\beta-1}$ is defined by (6.3). From (3.6) and (3.7), and the assumption that $T \times T$ is a conservative and ergodic map, we can obtain

$$(\mu \times \mu)(\varphi(x, y) \leq n) \sim \frac{1}{\Gamma(3-2\beta)\Gamma(2\beta)} \frac{n}{a'_n},$$

from which $(\mu \times \mu)(\varphi(x, y) \leq n) \in RV_{2(1-\beta)}$ follows.

Now, we have

$$\begin{aligned} E \left| \sum_{k=1}^n f_k(V_i) f_{k+h}(V_j) q(V_i)^{-1/\alpha'} q(V_j)^{-1/\alpha'} \right|^{\alpha'} &= \int_{E \times E} |S_n(f \times f)(x, y)|^{\alpha'} (\mu \times \mu)(dx \, dy) \\ &= (a'_n)^{\alpha'} (\mu \times \mu)(\varphi(x, y) \leq n) \int_{E \times E} \left| \frac{S_n(f \times f)(x, y)}{a'_n} \right|^{\alpha'} (\mu \times \mu)_n(dx \, dy). \end{aligned}$$

Uniform integrability of $(|S_n(f \times f)/a'_n|^{\alpha'}, n \geq 1)$ and (6.10) guarantee

$$\int_{E \times E} \left| \frac{S_n(f \times f)(x, y)}{a'_n} \right|^{\alpha'} (\mu \times \mu)_n(dx \, dy) \rightarrow \mu(f)^{2\alpha'} \Gamma(2\beta)^{\alpha'} EM_{2\beta-1}(1 - V_{2\beta-1})^{\alpha'} < \infty.$$

On the other hand, (4.5) implies $c_n^{\alpha'} \in RV_{\alpha'(\beta+2(1-\beta)/\alpha)}$. Thus,

$$E \left| \frac{1}{c_n} \sum_{k=1}^n f_k(V_i) f_{k+h}(V_j) q(V_i)^{-1/\alpha'} q(V_j)^{-1/\alpha'} \right|^{\alpha'} \in RV_{(2\beta-1)\alpha' + 2(1-\beta) - \alpha'(\beta+2(1-\beta)/\alpha)}.$$

Owing to the constraint on ξ , we have $(2\beta - 1)\alpha' + 2(1 - \beta) - \alpha'(\beta + 2(1 - \beta)/\alpha) < 0$ and hence (6.9) is complete. \square

The argument in Lemma 6.5 requires that the product map $T \times T$ to be conservative and ergodic. However, this is not necessarily true in general; see Example 5.2. It is, therefore, beneficial to get (6.9) without conservativity and ergodicity of the product map $T \times T$.

Lemma 6.6. *Suppose the conditions of Theorem 4.1, where $0 < \alpha \leq 1$ and $1/(2 - \alpha) \leq \beta < 1$, and particularly assume condition (ii). We fix $0 < \xi < \alpha^2/(\alpha + 2)$ and let $\alpha' = \alpha - \xi$. Then, (6.9) follows.*

Proof. We start by claiming

$$(6.12) \quad \frac{1}{a'_n} \sum_{k=1}^n (\widehat{T \times T})^k \mathbf{1}_{A \times A}(x, y) \rightarrow \mu(A)^2 \quad \text{uniformly, a.e. on } A \times A,$$

where

$$a'_n = \left(\frac{\Gamma(1 + \beta)}{\Gamma(\beta)} \right)^2 \frac{\Gamma(2\beta - 1)}{\Gamma(2\beta)} \frac{a_n^2}{n}.$$

Indeed, from (3.8) and (4.11), we see that

$$\begin{aligned} \sum_{k=1}^n (\widehat{T \times T})^k \mathbf{1}_{A \times A}(x, y) &= \sum_{k=1}^n \widehat{T}^k \mathbf{1}_A(x) \widehat{T}^k \mathbf{1}_A(y) \\ &\sim \frac{\mu(A)^2}{\Gamma(\beta)^2 \Gamma(2-\beta)^2} \sum_{k=1}^n \frac{1}{w_k^2} \quad \text{uniformly, a.e. on } A \times A. \end{aligned}$$

Applying Karamata's Tauberian theorem for power series to relation (3.7),

$$\sum_{k=1}^n \frac{1}{w_k^2} \sim \Gamma(2-\beta)^2 \Gamma(1+\beta)^2 \frac{\Gamma(2\beta-1)}{\Gamma(2\beta)} \frac{a_n^2}{n} \quad \text{as } n \rightarrow \infty.$$

Thus, (6.12) is obtained.

Now, (6.12) ensures that $A \times A$ can be viewed as a Darling-Kac set for the product map $T \times T$. As argued in Remark 6.3, even if $T \times T$ is neither conservative nor ergodic,

$$\int_{E \times E} \left(\frac{S_n(\mathbf{1}_{A \times A})(x, y)}{a'_n} \right)^r (\mu \times \mu)(dx \, dy) \sim \mu(A)^{2r} r! \frac{\Gamma(2\beta)^{r-1}}{\Gamma(r(2\beta-1) + 3 - 2\beta)} \frac{n}{a'_n}$$

holds.

Next, we define a probability measure $(\mu \times \mu)_n(\cdot)$ by

$$(\mu \times \mu)_n(\cdot) = (\mu \times \mu)(\{\varphi(x) \leq n, \varphi(y) \leq n\} \cap \cdot) / \mu(\varphi \leq n)^2.$$

Notice that the above definition of $(\mu \times \mu)_n$ differs from (6.11) given in Lemma 6.5. Then, we have

$$\begin{aligned} \int_{E \times E} \left(\frac{S_n(\mathbf{1}_{A \times A})(x, y)}{a'_n} \right)^r (\mu \times \mu)_n(dx \, dy) &= \frac{1}{\mu(\varphi \leq n)^2} \int_{E \times E} \left(\frac{S_n(\mathbf{1}_{A \times A})(x, y)}{a'_n} \right)^r (\mu \times \mu)(dx \, dy) \\ &\rightarrow \mu(A)^{2r} r! \frac{\Gamma(\beta)^2 \Gamma(2-\beta)^2 \Gamma(2\beta)^r}{\Gamma(2\beta-1) \Gamma(r(2\beta-1) + 3 - 2\beta)} \equiv \mu(A)^{2r} \eta_r. \end{aligned}$$

The sequence (η_r) determines, uniquely in law, a random variable Z_β , whose r th moment coincides with η_r itself. To see this, it is enough to check the Carleman sufficient condition $\sum_{k=1}^{\infty} \eta_{2k}^{-1/2k} = \infty$, which can be easily checked by Stirling's formula together with elementary algebra. It is thus concluded that with respect to $(\mu \times \mu)_n$,

$$\frac{S_n(\mathbf{1}_{A \times A})(x, y)}{a'_n} \Rightarrow \mu(A)^2 Z_\beta \quad \text{in } \mathbb{R}.$$

Since f is a bounded and is supported by A , there is a constant $C_1 > 0$ such that

$$\begin{aligned} E \left| \sum_{k=1}^n f_k(V_i) f_{k+h}(V_j) q(V_i)^{-1/\alpha'} q(V_j)^{-1/\alpha'} \right|^{\alpha'} &= \int_{E \times E} |S_n(f \times f)(x, y)|^{\alpha'} (\mu \times \mu)(dx \, dy) \\ &\leq C_1 \int_{E \times E} |S_n(\mathbf{1}_{A \times A})(x, y)|^{\alpha'} (\mu \times \mu)(dx \, dy) \\ &= C_1 (a'_n)^{\alpha'} \mu(\varphi \leq n)^2 \int_{E \times E} \left| \frac{S_n(\mathbf{1}_{A \times A})(x, y)}{a'_n} \right|^{\alpha'} (\mu \times \mu)_n(dx \, dy). \end{aligned}$$

Because of the uniform integrability of $(|S_n(\mathbf{1}_{A \times A})/a'_n|^{\alpha'}, n \geq 1)$, we see that $\int_{E \times E} |S_n(\mathbf{1}_{A \times A})/a'_n|^{\alpha'} d(\mu \times \mu)_n$ converges to some positive finite constant. The rest of the discussion is the same as Lemma 6.5. \square

Finally, we provide useful inequalities that will supplementally be used in the proof of Propositions 4.5 and 4.7.

Lemma 6.7. *Fix $\xi > 0$ as specified in Lemma 6.4, 6.5, or 6.6. Let $\alpha' = \alpha - \xi$ and define*

$$W_{ij}^{(n, \alpha')} = \frac{1}{c_n} \sum_{k=1}^n f_k(V_i) f_{k+h}(V_j) q(V_i)^{-1/\alpha'} q(V_j)^{-1/\alpha'}.$$

Let

$$\ln_+ x = \begin{cases} \ln x & \text{if } x > 1, \\ 0 & \text{otherwise.} \end{cases}$$

(a) *There exist an integer $m_0 > 0$ and constants $C > 0$, $\gamma < \alpha'$, such that for any $m \geq m_0$,*

$$\begin{aligned} E \left| \sum_{m < i < j < \infty} \epsilon_i \epsilon_j U_{\alpha,1}^{\leftarrow} \left(\frac{\Gamma_i q(V_i)}{2} \right) U_{\alpha,1}^{\leftarrow} \left(\frac{\Gamma_j q(V_j)}{2} \right) \frac{1}{c_n} \sum_{k=1}^n f_k(V_i) f_{k+h}(V_j) \mathbf{1}_{\{|W_{ij}^{(n, \alpha')}|^{\alpha'} \leq ij\}} \right|^{\alpha'} \\ \leq C \left(E(|W_{ij}^{(n, \alpha')}|^{\alpha'} (1 + \ln_+ |W_{ij}^{(n, \alpha')}|)) \right)^\gamma, \\ E \left| \sum_{m < i < j < \infty} \epsilon_i \epsilon_j U_{\alpha,1}^{\leftarrow} \left(\frac{\Gamma_i q(V_i)}{2} \right) U_{\alpha,1}^{\leftarrow} \left(\frac{\Gamma_j q(V_j)}{2} \right) \frac{1}{c_n} \sum_{k=1}^n f_k(V_i) f_{k+h}(V_j) \mathbf{1}_{\{|W_{ij}^{(n, \alpha')}|^{\alpha'} > ij\}} \right|^{\alpha'} \\ \leq CE(|W_{ij}^{(n, \alpha')}|^{\alpha'} (1 + \ln_+^2 |W_{ij}^{(n, \alpha')}|)). \end{aligned}$$

(b) *There exist an integer $m_0 > 0$ and constants $C > 0$, $\gamma < \alpha'$, such that for any $m \geq m_0$ and $i \geq 1$,*

$$\begin{aligned} E \left| \sum_{j=m+1}^{\infty} \epsilon_j U_{\alpha,1}^{\leftarrow} \left(\frac{\Gamma_j q(V_j)}{2} \right) \frac{1}{c_n} \sum_{k=1}^n f_k(V_i) f_{k+h}(V_j) q(V_i)^{-1/\alpha'} \mathbf{1}_{\{|W_{ij}^{(n, \alpha')}|^{\alpha'} \leq j\}} \right|^{\alpha'} \leq C(E|W_{ij}^{(n, \alpha')}|^{\alpha'})^\gamma, \\ E \left| \sum_{j=m+1}^{\infty} \epsilon_j U_{\alpha,1}^{\leftarrow} \left(\frac{\Gamma_j q(V_j)}{2} \right) \frac{1}{c_n} \sum_{k=1}^n f_k(V_i) f_{k+h}(V_j) q(V_i)^{-1/\alpha'} \mathbf{1}_{\{|W_{ij}^{(n, \alpha')}|^{\alpha'} > j\}} \right|^{\alpha'} \\ \leq CE(|W_{ij}^{(n, \alpha')}|^{\alpha'} (1 + \ln_+ |W_{ij}^{(n, \alpha')}|)). \end{aligned}$$

Proof. The proof is analogous to that of Proposition 5.1 in Samorodnitsky and Szulga (1989), but an obvious upper bound $U_{\alpha,1}^{\leftarrow}(x) \leq Cx^{-1/\alpha'}$, $x > 0$ has to be suitably applied. \square

Remark 6.8. The inequalities in Lemma 6.7 will still hold, even if the parameter α' and the inverse function $U_{\alpha,1}^{\leftarrow}(\cdot)$ are replaced by the constant p_0 given in (4.1) and $U_{\alpha,2}^{\leftarrow}(\cdot)$, respectively.

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REFERENCES

- AARONSON, J. (1981): “The asymptotic distributional behaviour of transformations preserving infinite measures,” *Journal d’Analyse Mathématique*, 39, 203–234.
- (1997): *An Introduction to Infinite Ergodic Theory*, volume 50 of *Mathematical Surveys and Monographs*, Providence: American Mathematical Society.
- BINGHAM, N. (1971): “Limit theorems for occupation times of Markov processes,” *Probability Theory and Related Fields*, 17, 1–22.
- BINGHAM, N., C. GOLDIE, AND J. TEUGELS (1987): *Regular Variation*, Cambridge: Cambridge University Press.
- BROCKWELL, P. AND R. DAVIS (1991): *Time Series: Theory and Method*, 2nd edition, New York: Springer.
- DAVIS, R. AND T. MIKOSCH (1998): “The sample autocorrelations of heavy-tailed processes with applications to ARCH,” *The Annals of Statistics*, 26, 2049–2080.
- DAVIS, R. AND S. RESNICK (1986): “Limit theory for the sample covariance and correlation functions of moving averages,” *The Annals of Statistics*, 14, 533–558.
- (1996): “Limit theory for bilinear processes with heavy-tailed noise,” *The Annals of Applied Probability*, 6, 1191–1210.
- HARRIS, T. AND H. ROBBINS (1953): “Ergodic theory of Markov chains admitting an infinite invariant measure,” *Proceedings of the National Academy of Sciences of the United States of America*, 39, 860–864.
- KALLENBERG, O. (1997): *Foundations of Modern Probability*, 1st edition, New York: Springer.
- KESSEBÖHMER, M. AND M. SLASSI (2007): “Limit laws for distorted critical return time processes in infinite ergodic theory,” *Stochastics and Dynamics*, 7, 103–121.
- KRENGEL, U. (1985): *Ergodic Theorems*, Berlin: de Gruyter.
- PRATT, J. (1960): “On interchanging limits and integrals,” *Annals of Mathematical Statistics*, 31, 74–77.
- RAJPUT, B. AND J. ROSÍŃSKI (1989): “Spectral representations of infinitely divisible processes,” *Probability Theory and Related Fields*, 82, 451–488.
- RESNICK, S. (1987): *Extreme Values, Regular Variation and Point Processes*, New York: Springer-Verlag.

- RESNICK, S., G. SAMORODNITSKY, AND F. XUE (1999): “How misleading can sample ACFs of stable MAs be? (Very!),” *The Annals of Applied Probability*, 9, 797–817.
- (2000): “Growth rates of sample covariances of stationary symmetric α -stable processes associated with null recurrent Markov chains,” *Stochastic Processes and Their Applications*, 85, 321–339.
- RESNICK, S. AND E. VAN DEN BERG (2000): “Sample correlation behavior for the heavy tailed general bilinear process,” *Stochastic Models*, 16, 233–258.
- ROSIŃSKI, J. (1990): “On series representations of infinitely divisible random vectors,” *The Annals of Probability*, 18, 405–430.
- ROSIŃSKI, J. AND G. SAMORODNITSKY (1993): “Distributions of subadditive functionals of sample paths of infinitely divisible processes,” *The Annals of Probability*, 21, 996–1014.
- ROSIŃSKI, J. AND T. ŽAK (1996): “Simple conditions for mixing of infinitely divisible processes,” *Stochastic Processes and Their Applications*, 61, 277–288.
- ROY, E. (2008): “Ergodic properties of Poissonian ID processes,” *The Annals of Probability*, 35, 551–576.
- SAMORODNITSKY, G. (2004): “Extreme value theory, ergodic theory, and the boundary between short memory and long memory for stationary stable processes,” *The Annals of Probability*, 32, 1438–1468.
- (2005): “Null flows, positive flows and the structure of stationary symmetric stable processes,” *The Annals of Probability*, 33, 1782–1803.
- SAMORODNITSKY, G. AND J. SZULGA (1989): “An asymptotic evaluation of the tail of a multiple symmetric α -stable integral,” *The Annals of Probability*, 17, 1503–1520.
- SAMORODNITSKY, G. AND M. TAQQU (1994): *Stable Non-Gaussian Random Processes*, New York: Chapman and Hall.
- THALER, M. (2000): “The asymptotics of the Perron-Frobenius operator for a class of interval maps preserving infinite measures,” *Studia Mathematica*, 143, 103–119.
- (2001): *Infinite ergodic theory*, The Dynamic Odyssey course, CIRM.
- THALER, M. AND R. ZWEIMÜLLER (2006): “Distributional limit theorems in infinite ergodic theory,” *Probability Theory and Related Fields*, 135, 15–52.
- ZWEIMÜLLER, R. (2000): “Ergodic properties of infinite measure-preserving interval maps with indifferent fixed points,” *Ergodic Theory and Dynamical Systems*, 20, 1519–1549.
- (2004): “ S -unimodal Misiurewicz maps with flat critical points,” *Fundamenta Mathematicae*, 181, 1–25.

- (2007): “Infinite measure preserving transformations with compact first regeneration,” *Journal d’Analyse Mathématique*, 103, 93–131.
- (2009): *Surrey Notes on Infinite Ergodic Theory*, Lecture notes, Surrey University.

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